Mathematics and Narrative

Part One: Mathematics is the muse of everything

For over two millennia, it has seemed irresistible to imagine that the essence of all reality lies in numbers, and that a transcendent system of mathematics, if we could only know it synoptically, cosmically, would provide the answers to our questions. The Pythagorean insight that the capacious and sensory world of music is an avatar of ratios of numbers, and that, as a result, melody and harmony carry mathematical patterns, must have seemed like a revelation.

The idea that mathematics lies somehow behind everything has motivated inquiry since the Ancient Greeks. If music—from Pythagoras to fractals—can be mathematics, and the heavens and their motions can be mathematics, perhaps human biology and even the soul are mathematical systems, to be tuned like stringed instruments to their perfect order.

This impulse to seek the essence of reality in mathematics is compelling. As an undergraduate student, I hoped to find within mathematics some technical instruments sufficient to describe the human central nervous system and the brain. The impulse to do so was not mystical or merely reductionistic. The human brain has on the order of 10 to 100 billion (10^{11}) neurons. The average number of synaptic connections per neuron is on the order of ten thousand (10^4). The total number of connections in the brain is therefore maybe ten trillion to a quadrillion (10^{13}-10^{15}). To a mathematician, these numbers are just integers, but to a physicist or a neurobiologist, they are huge. 10^{15} connections is about ten thousand times as many stars as astronomers think might be in the entire Milky Way galaxy (10^{11}). All those connections, inside your head, in a system weighing about 1.4 kilograms. The timing and phases of firing in neuronal groups, the suites of ontogenetic development in the brain, the electrochemical effect of neurotransmitters on receptors, the scope and mechanisms of neurobiological plasticity, the regulation of genetic expression in neuroembryology—all invite mathematical treatment. But where is the mathematics sufficient to capture and express such a vast dynamical system, with its very large scale integration? If the brain develops according to darwinian principles, where is the mathematics sufficiently powerful to model that descent? This question returned to me when I encountered Gerald Edelman's exploration of the thesis that the brain is a somatic selection system.

The ambition to deploy mathematics to help explain human systems has reached in many directions. It has reached to languages in mathematical linguistics, with its algebraic flavor, and also in formal linguistics, which, although it rarely uses principles of mathematical systems, nonetheless imitates their notational trappings.

Can mathematics show us something about narrative? The question has seemed plausible to many scholars. After all, it is connected to both music and language, which carry narrative patterns, through dynamics and clausal forms. Accordingly, many traditions—from Vladimir Propp's *Morphology of the Folktale* through the French and Russian structuralists, formalists, and semioticians—analyze narrative structure as quasi-algebraic systems, with elements, relations, sequences, and constraints on their combination. What sequences are possible? How do they recur? Which are basic?

Geometric concepts are also routinely applied to narrative, as when we hear of the necessary *arc* of character development. Madison Smart Bell, in his chapter on "Linear Design" in *Narrative Design: A Writer's Guide to Structure,* explains that "There are many possible structures for a narrative, but the most common, familiar, and conventional of these is linear design. Linear stories start at the beginning, traverse some sort of middle, and stop at the end. Furthermore, all linear designs bear some relationship to what is known as the Freitag triangle" (p. 27). The Freitag triangle is an isosceles triangle with a base whose vertices are exposition and resolution, and another vertex in the middle that represents a climax.

More recently, artificial intelligence programmers, principally in the 1970s and 1980s, borrowed from structuralist, formalist, and semiotic traditions to program computers with story grammars, for the purpose of parsing connected natural texts such as newspaper stories. These computer programs have been deployed to generate texts that human beings can interpret as stories.

These are some of the ways in which mathematics has been used to approach narrative.

Part Two: Tell me a story

Narrative has equally been used to approach mathematics.

First, of course, stories often include mathematical content, especially in science fiction and fantasy, but in fact mathematics shows up in nearly every kind of narrative, even humorous narratives. In *The Hitchhiker's Guide to the Galaxy,* Douglas Adams treats us to a disquisition on the nature of "bistromath" and its role in the invention of the infinite improbability drive.

Second, stories are a tried and true mechanism for teaching children about mathematics. "Word problems"—almost universally despised by children as torture—pose mathematical problems by means of small narratives of tinkers, tailors, soldiers, spies, cannibals and missionaries, and prisoners caught in dilemmas. *The Phantom Tollbooth*, by Norton Juster, provides many universally-loved stories that teach mathematical concepts.

Third, narrative provides a great service to the world of mathematics by conveying to those who are not engaged in mathematics what it might be like to inhabit that world. This is one of the great virtues of Apostolos Doxiadis's *Uncle Petros and Goldbach's Conjecture*. Who knows how many young people will become interested in mathematics and logic by reading about Manga the dog and Bertrand Russell's childhood in *Logicomix*?

Part Three: Story is the muse of mathematics

Here I will explore quite a different relationship between narrative and mathematics: Story—the sophisticated human mental ability to conceive of orchestrated suites of events, objects, agents, and actions—is much older than any sophisticated mathematical accomplishment. Our advanced abilities for mathematics are based in part on our prior cognitive ability for story. There are basic human cognitive operations that make it possible for us to invent mathematical concepts and systems. One of those operations is the fundamental human operation of *story*—of understanding the world and our agency in it through certain kinds of human-scale conceptual organizations involving agents and actions in space. Another basic human cognitive operations that makes it possible for us to invent mathematical concepts and systems is "conceptual integration," also called "blending." *Story* and *blending* work as a team.

Here, I will explore how the operations of story (or narrative thinking, if you like) and blending combine to give us abilities for mathematical thought.

Blending and the Human Mind

What is blending? (Technical introductions to the nature and mechanisms of blending can be found in Fauconnier and Turner, 2002 and 1998; Fauconnier 1997; Turner 2001 and 2005. See also Goguen 1999.)

Let us begin with an example. A man is participating in a wedding. He is consciously enacting a familiar mental story, with roles, participants, a plot, and a goal. But while he is fulfilling his role in the wedding story, he is remembering a different story, which took place a month before off the Cycladic island of Despotico, where he and his girlfriend, who is not present at the wedding, went diving in the hopes of retrieving sunken archeological treasures from the newly-discovered Temple of Apollo and Artemis. Why, cognitively, should he be able mentally to activate these two stories simultaneously? There are rich possibilities for confusion, but in all the central ways, he remains unconfused. He does not mistake the bride for his girlfriend, for the treasure, for the fish, for the temple, or even for Artemis. He does not swim down the aisle or speak as if through a snorkel. He does not think that the wedding ring is a recovered archeological treasure.

Now notice something interesting. Human beings go beyond merely imagining stories or concepts that run counter to the present environment. We can also connect them and *blend* them to make a third mental array. The man at the wedding can make analogical connections between his girlfriend and the bride and between himself and the groom, and blend these counterparts into a daydream in which it is he and his girlfriend who are being married at this particular ceremony. This blended story is manifestly false, and he should not make the mistake, as he obediently discharges his duties at the real wedding, of thinking that he is in the process of marrying his girlfriend. But he can realize that he likes the blended story, and so formulate a plan of action to make it real. Or, in the blended story, when the bride is invited to say ‘I do,’ she might say, ‘I would never marry you!’ Her fulguration might reveal to him a truth he had sensed intuitively but not recognized.

Blending, Small Spatial Stories, and Mathematics

Now for mathematics. Consider the concept of a *proof*. A proof is a rigorous mathematical argument demonstrating a thesis, but how do we conceive of an argument in the first place?

In *The Literary Mind* (1996), I discuss how human beings are especially equipped to understand the world by making mental blends that are based on small spatial stories. We are very good at thinking in terms of small spatial stories. We are built for it, and we are built to use small spatial stories as inputs to conceptual blends. In small spatial stories, we separate events from objects and think of some of those objects as actors who perform physical and spatial actions. We routinely understand our worlds by constructing a conceptual integration network in which one of the inputs is a small spatial story.

For example, consider an event that has no actors or actions. We can blend it with a small spatial story that does have an intentional actor and an action, to achieve a human scale small spatial story. In the blend, the event becomes an action by an actor: the wind beats the sea; the waves swallow the boat; the boulder refuses to be moved; the car decides to stop moving.

In these small spatial stories, actors are self-propelled movers across a landscape, and they manipulate objects. There is a strong overlap and supplementarity in our experience between stories of self-propelled movement and stories of manipulation of physical objects. To manipulate an object, we often must go to it, move our arm and hand toward it, grasp the object, and manipulate it. Someone who is "going for an object" is usually moving his entire body in the direction of the object, moving his hands toward the object, and intending to grab the object and manipulate it, perhaps by putting it in his pocket. If we say of an act of theft, "One man *ran up to* another man to *snatch the jewel away from him* so he could *run away with it*," we have an example of the overlap of a small spatial story of moving with a small spatial story of manipulation. It is no surprise that we call intentional actors "movers and shakers."

Eve Sweetser (1990) has examined the elaborate system according to which we connect action stories of movement with stories of thinking. She calls this conceptual system THE MIND IS A BODY MOVING THROUGH SPACE. I analyze this system as the product of conceptual blending. THE MIND IS A BODY MOVING THROUGH SPACE is a special case of blending human activity with stories of intentional actors moving and manipulating.

For example, we can say of a chess match, "Experts thought that white would *take* the draw, but white's next move made it clear that white was *heading for* a win." Here, we have a blend of playing chess (as a mental activity, not as moving pieces on a board) with a small spatial story of manipulating and moving (taking and moving). Through blending, actors are "movers and shakers" regardless of whether their actions are spatial.

When we wish to tell the action-story of a mathematical discovery, we can say that the thinker *began from* a certain assumption, was *headed for* a certain conclusion, *stumbled over* difficulties, *moved faster* or *slower* at various times, had to *backtrack* to correct mistakes, *obtained part of* the solution but was still *missing* the most important *part*, had a notion of *where to look for it*, began at last to *see* it, *followed it* as it *eluded* her, finally *got one finger on it*, *felt it slip nearly away*, but at last *got it*. Of course, after she has made the discovery, it becomes *hers*. This is a case in which an actor in a non-spatial story of thinking is understood by blending it with a spatial action-story of moving and manipulating.

We can blend our notion of *thought* with small spatial stories of physical manipulation that do not involve self-propelled locomotion. In this blend, the actor only manipulates objects as instruments, tools, or aids to fabrication. When we talk of cognitive "instruments" or conceptual "tools" or of "piecing together a proof," we are blending the activity of thought with the action of manipulation, specifically manipulation for the purpose of manufacture. We may "apply" a theorem in the way we "apply" a template. We may "carve" out a new mathematical concept in the way we "carve" a statue out of wood or stone.

Small spatial stories of moving and manipulating go together in our experience. We conceive of thought largely by blending our notion of mental activity with small spatial stories of actors as movers and manipulators.

Now let us consider the special case of a quest story. A quest story can be a story of actual spatial movement and manipulation. A *blended quest story* is one in which non-spatial activity is blended with a quest story of spatial movement and manipulation. For example, we can blend our conception of planning, conniving, and reasoning with a small spatial quest story to achieve the blend of a *mental quest*. If the mental activity is mathematical, this produces the special case of a *mathematical quest*. And if the mathematical mental activity is the action of making our way through a mathematical proof, then we have the yet more special case of a *proof quest*. In this special case, the thinking is mathematical, and the mathematician is a quester. Apostolos Doxiadis, in "Euclid's Poetics: An examination of the similarity between narrative and proof," a lecture given at the Mathematics and Culture conference in Venice in April 2001, points out, quite correctly, that spatial conceptions of locations, paths, objects, and intentional movement and manipulation are at the base of our conception of proving a theorem. I would say, in my terms, that Doxiadis is exploring the *mental quest* blend, with emphasis on the special case of the *mathematical mental quest* and the yet more special case of the *proof quest*.

Given human experience, there is no more compelling story than a blended quest story. As Apostolos Doxiadis writes in "The mathematical logic of narrative,"

> Some of the basic axioms of this Hollywood cosmology of film are:
>
> 1. All stories are really quest stories.

1.1 If a story does not appear to be a quest story at first glance, delve deeper into it and unearth the quest story within it.

1.2 If you fail in 1.1 drop the story, it ain't worth it.

2. All quest stories are about a sympathetic hero searching for, and finding, a treasure.

2.1 The treasure may be material (money, object, secret weapon, world cup) or immaterial (love, salvation, knowledge, etc.)

3. The interest of the story is determined by certain factors, among them: a) how important the treasure is, b) how badly the hero or heroine wants it, c) how difficult the quest is. The more difficult, the more tickets at the box office.

4. The difficulty of the journey is incarnate in a person, called the Antagonist.

And so on.

In a spatial story of a quest, the intentional actor is literally moving from spatial location to spatial location, aided or impeded by geography, elements, or other actors. When we blend this story with the mathematical activity of proving a theorem, the completion of the proof is, in the blend, the object of the quest; the mathematician is the quester; the object-theorem is a blended location; the quester moves/proceeds step-by-step; whatever step he has reached in the proof is his current location; he continues to seek paths and avenues to the next step; sometimes he must abandon a path and return to a previous one; he can become stuck. There are somewhat different varieties of this blend. In a spatial quest story, the goal can be to obtain a particular object, or to arrive at a particular place, or to complete the entire journey, or indeed all of these—completing a journey to arrive at a place and so acquire the object in that place. Accordingly, in the *proof quest*, the goal can be to complete the entire proof, or to arrive at its final conclusion, or to possess, mentally, the truth of that final conclusion, or all of these things together. The reductio ad absurdum blended quest is treated in *Cognitive Dimensions of Social Science* (Turner 2001).

There are of course other blends based on small spatial stories that help us conceive of mathematical proof, namely, manipulation stories. The proof can be an object to be manufactured; the mathematician can try to piece it together from parts; he can try to connect one part to the other but be unable to tie it together.

Geometric Constructions

The human activity of proving a theorem is one thing, but what about the mathematical objects themselves, as opposed to the thought of the mathematician and the proofs that this thought generates? The hallmark of Greek geometry, for example, is stasis rather than dynamics. What could be the role of small spatial stories in a world with no movement, no change?

Parmenides described a world without small spatial stories in *The Way of Truth*. But as Parmenides indicates in *The Way of Seeming*, we do not think that way. For us, there are objects, events, causes, effects, change.

The conceptual roots of Greek geometry lie in small stories of manipulation, namely, of physical construction using a compass, or a compass and a straightedge. Geometric objects were defined by construction. The three great problems of classical antiquity were defined as such because they were objects without a story to produce them: squaring the circle, duplicating the cube, and trisecting the angle. Even simple algebraic operations were understood through construction: addition, subtraction, multiplying a number by a rational number, and so on. It is remarkable that mathematics would have placed such a responsibility for mathematical truth upon the existence of a small spatial story for producing the object.

Human Scale

There are many realms of mathematics for which it is obvious that the conceptual roots are in small spatial stories.

- Functions can be thought of as "carrying" one value into another, or as "taking" one or more values and "transforming" them into another value. In these conceptions, functions are movers and manipulators in small spatial stories.
- Numbers are conceived of through small spatial stories. One number is the "successor" of another in set theory. Numbers are defined through concepts such as "cuts," or "limits" that are "approached."
- Analysis is defined through small spatial stories. It is the study of "rates of change" of quantities.
- A derivative, in analysis, is defined in terms of coordinated changes, thought of as movements. It is perhaps most natural for human beings to think of a derivative as the rate of change of the value of a function over the value of *time*, the study of what Newton called "fluxions." But the parameter need not be time. The derivative is thought of intuitively as the rate of the change of the function with respect to the change of the value of the parameter.
- An integral, in analysis, can be thought of as a static area, but even that area is the result of an operation: we "integrate" over the values of a function as its parameter varies, to achieve the definite integral.

As long as such mathematical conceptions are based in small spatial stories at human scale, that is, fitting the kinds of scenes for which human cognition is evolved, mathematics can seem straightforward, even natural. The same is true of physics. If mathematics and physics stayed within these familiar story worlds, they might as disciplines have the cultural status of something like carpentry: very complicated and clever, and useful, too, but fitting human understanding.

The problem comes when mathematical work runs up against structures that do not fit our basic stories. In that case, the way we think begins to fail to grasp the mathematical structures. The mathematician is someone who is trained to use conceptual blending to achieve new blends that bring what is not at human scale, not natural for human stories, back into human scale, so it can be grasped. Let us look at a few different spots.

- Incommensurability. In the normal human scene, 1 is a cognitive reference point, and if we take any length as a unit, and make a cut, it seems to break into two parts, commensurate with the unit, that sum to make the unit, such as one-half and one-half. So what do you do when someone finds a number that is not commensurate with unity, like the square root of two? Well, according to legend, you throw him overboard and thereafter revert to the normal human scene of construction. Then you are safe: you can actually construct a hypotenuse of an isosceles right triangle with legs equal to unity, so relax; don't pursue unnatural directions. Mathematicians have worked around the difficulty by reforming the concept of number, as Fauconnier and I discuss in *The Way We Think,* to give a new human-scale blend that is not available to most people.
- Rates of change and limits. Plenty of people can understand the idea of a rate of change. It fits our world of human-scale small spatial stories. But asking them to think of limits that are approached but not reached does not fit our stories so well. We never approach something and keep moving without reaching it. We have all seen scores of bright freshmen fail to understand the simple definition of a derivative of a function as the limit

as h approaches zero of a quotient, where the divisor of the quotient is h and the numerator is the difference between the function of the quantity x + h and the function of x:

$$f'(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

There is additional difficulty when someone tries to understand that the derivative might not exist. How can there be no rate of change? For the derivative to exist, the limit must exist as it is approached from both "directions." How can something approach a limit from one direction but not from another? The existence of functions without derivatives has disturbed even superb mathematicians.

One of my favorite things to study when I was at Berkeley was counterexamples in mathematics. Many counterexamples are cases where a small spatial story provides a fine conceptual understanding for a mathematical system but leads to intuitions that are not sustained technically. So counterexamples are often surprising.

It is a great virtue of conceptual integration that we can create new blends at human scale, by using something that is already at human scale as one input to the blend and by performing various kinds of conceptual compression on structure within inputs and on relations between inputs.

Let me give an introduction to conceptual compression to human scale from the domain of language.

One of our most basic small spatial stories is Caused Motion: an agent performs an action that causes an object to move in a direction, as in "He threw the ball over the fence" (Goldberg 1995). This is associated in English with the Caused-Motion construction, whose basic form is NounPhrase-VerbPhrase-NounPhrase-PrepositionalPhrase. But we can make sense of many things that are not caused motion by blending them with the Caused Motion small spatial story, and even prompting for their understanding in the same clausal form:

- I walked him into the room.
- He sneezed the napkin off the table.
- I pointed him toward the door.
- They teased him out of his senses.
- I will talk you through the procedure.
- I read him to sleep.
- They prayed the two boys home.
- I muscled the box into place.
- Hunk choked the life out of him.
- He floated the boat to me.

And this kind of compression to human scale can be used for conceptual arrays that are not at human scale, so

- The Upper Paleolithic brought new opportunities to humankind.

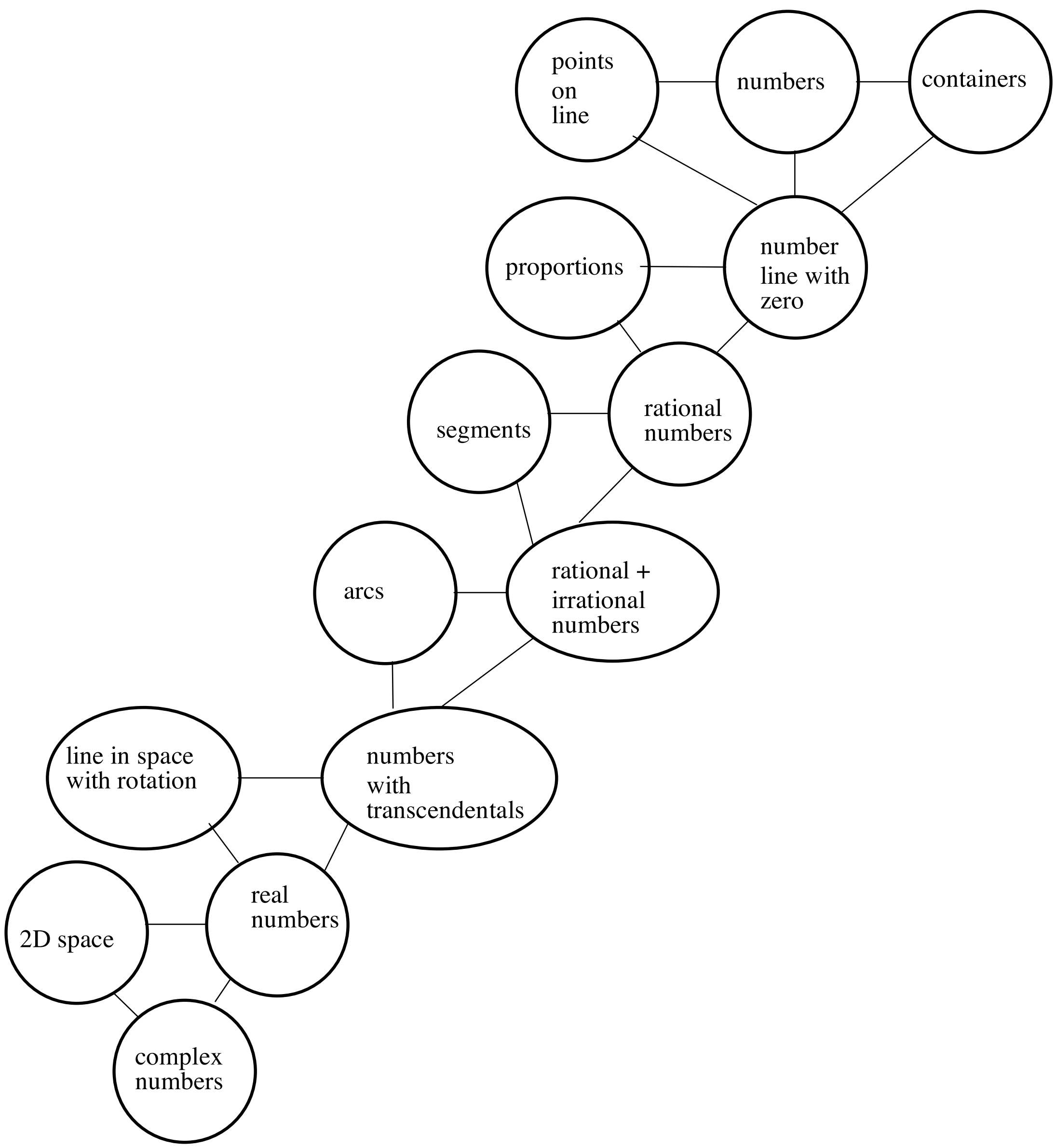
Similarly, consider the Resultative construction in English, which has the form NounPhrase-Verb-NounPhrase-Adjective, where the Adjective denotes a property C (Goldberg 1995). This clausal form carries a small story: *A does something to B with the result that B has property C*, as in "Kathy painted the wall white." We can make blends that are organized by the Resultative story, and this gives human narrative scale to many things:

- Last night's meal made me sick.
- I boiled the pan dry.
- The earthquake shook the building apart.
- Roman imperialism made Latin universal.

Each of these scenes has complicated structure, but all can be understood through the resultative story. The simple resultative story is at human scale and we can render complexities immediately graspable by blending them with the resultative story. The resultative story allows us to compress over Identity (e.g. Roman imperialism), Time, Space, Change, Cause-Effect, and Intentionality.

Complex Numbers

The same sort of compression to human scale through small spatial stories plays a role in mathematics. First, a brief history of how numbers arise through blending, following *The Way We Think*:



We start with three input mental spaces: whole numbers, discrete points on a line (such as first step, second step, third step in walking); and containers of objects. There is a counterpart mapping across these spaces connecting the steps, the whole numbers, and the containers. Additionally, the starting point on the path is mapped to the empty container. However, the empty container and the starting point on the path have no counterpart in the mental space with whole numbers. Blending these three inputs produces the number/line blend, with projection of the starting point and empty container to the blend, yielding there the emergent number zero. The operation of addition in the blend comes from the input with containers. In the blend, we have as emergent structure the addition of segments (with lengths 1, 2, 3, etc) by juxtaposition. We also have the emergent notion of a segment of zero length, which fits into the addition scheme. The idea of a segment of zero length is counterintuitive at first but acquires human scale in the blend, where it is just another segment. In the blend, the number line has numbers on only points 0, 1, 2, . . .

The next blend has as inputs the number line with zero and our concept of proportions to achieve rational numbers. This mental blend is altogether more complicated than one might at first imagine. Fauconnier and I analyze it at length in *The Way We Think*.

The next mental blend is commonly associated with Pythagoras. It blends together rational numbers and line segments. There is a simple cross-space mapping from segments to rational numbers since rational numbers are already on a line in virtue of previous blends. It turns out that some segments, such as the hypotenuses of some right triangles, have no counterpart in the number input. But in the blended space, those segments with one extremity at zero are already projected. In the blend, the other extremity is a number/ point. This gives the emergent notion in the blend of an irrational number. It follows that segments can have irrational length. Operations of polynomial roots, e.g. square roots, emerge as general operations in the blended space. In this blended space, every number now has an nth root.

Geometric blends allow one to view arc lengths as limits of added segment lengths. The history and mathematical details are complicated, but in principle this works like the previous blend: arcs are projected to arc length/numbers in the blended space yielding emergent numbers like π.

In the next step, there is a number input, which has a half-line, a ray. But the geometric input has a full line in 2D space. Mirror image points can be obtained by rotation by 180 degrees. So in the blended space, we have negative numbers on the line through 180-degree rotation from positive numbers. Addition is emergent as a geometric operation on segments: rotate from the extremity of segment 1 around the origin of segment 2 (by either 0 degrees or 180 degrees). Only the line is projected from the 2D space input.

Through successive operations of blending, the original concept of number as whole number, which is already very much at human scale, has been extended to produce blended concepts that are at human scale only because of the blending. The original system of whole numbers was extended to include zero and fractions. Some of this blending is very complicated, but once it works, after the fact, it looks as if new elements have been simply added to old ones, because we still use the same words for them. In fact, in the metamorphosis of the category *number*, the entire structure and organizing principles have been dramatically altered. It is an illusion that the old input is simply transferred wholesale as a subset of the new category.

The blending step I want to concentrate on is the one that produces complex numbers. Fauconnier and I review the history of this conceptual development, which was slow and fraught with difficulty. Square roots of negative numbers had shown up in formulas of sixteenth-century mathematicians and they had correctly formulated operations on these numbers. The square roots of negative numbers lent themselves to formal manipulations without fitting into a mathematical conceptual system. A genuine concept of complex number took time to develop.

This development begins from the number line blend. It was extended by Descartes to create the coordinate plane. The seventeenth-century mathematician John Wallis then observed in his 1685 book *Algebra* that if negative numbers could be mapped onto a directed line, complex numbers could be mapped onto points in a two-dimensional plane. He provided geometric constructions for the counterparts of the real or complex roots of $ax^2 + bx + c = 0$ (Kline 1980, page 272). In effect, Wallis provided a consistent model for the mysterious numbers, giving some substance to their formal manipulation. Although Wallis's mapping showed the formal consistency of a system including complex numbers, this was not enough to extend the concept of number. As Morris Klinereports, Wallis's work was ignored: it did not make mathematicians receptive to the use of such numbers. This is an interesting point in itself. Mapping a coherent space onto a conceptually incoherent one is not enough to give the incoherent space new conceptual structure. It also follows that coherent abstract structure is not enough, even in mathematics, to produce satisfactory conceptual structure: In Wallis's representation, the metric geometry provided abstract schemas for a unified interpretation of real and imaginary numbers, but this failed to persuade mathematicians to revise their domain of numbers accordingly. Only after the new conceptual structure of *complex number* develops in the blended space is the domain of numbers actually extended.

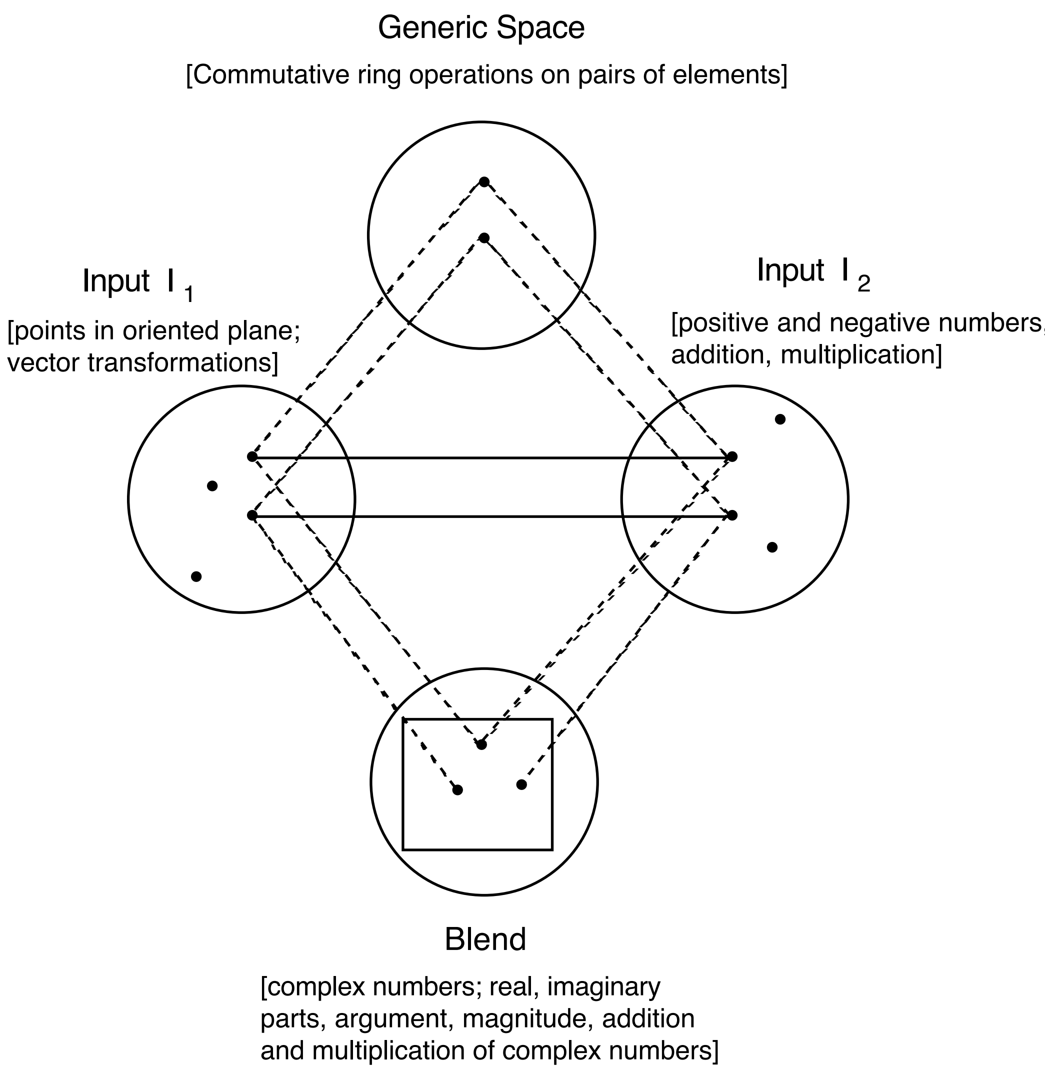
In this blend, but not in the original inputs, it is possible for an element to be simultaneously a number and a geometric point, with Cartesian coordinates (a, b) and polar coordinates (ρ, θ). In the blend, numbers have interesting general formal properties, such as

$$(a, b) + (a', b') = (a+a', b+b')$$

$$(\rho, \theta) \times (\rho', \theta') = (\rho\rho', \theta + \theta')$$

Every number in this extended sense has a real part, an imaginary one, an angle, and a magnitude. By virtue of the link of the blend to the geometric input space, the numbers can be manipulated geometrically; by virtue of the link of the blend to the input space of real numbers, the new numbers in the blend are immediately conceptualized as an extension of the old numbers.

The entire conceptual integration network has two inputs, two-dimensional geometric space and real numbers. Complex numbers and their properties emerge in the blended space. As in Wallis's scheme, the mapping from points on a line to numbers has been extended to a mapping from points in a plane to numbers. This mapping is partial from one input to the other—only one line of the plane is mapped onto the real numbers in the other input—but it is total from the geometric input to blend: all the points of the plane have counterpart complex numbers. And this in turn allows the blend to incorporate the full geometric structure of the geometric input space.



When a rich blended space of this sort is built, an abstract generic space comes along with it. Having the three spaces containing respectively points (input 1), numbers (input 2), complex point/numbers (blend) entails a fourth space with abstract elements having the properties "common" to points and numbers. The abstract notions in this case are "operations" on elements. For numbers, the operations are addition and multiplication. For points in the plane, the operations can be viewed as geometric transformations, like rotation and stretching. In the blended space of complex numbers,

numbers and vectors are the same thing, and so addition of numbers just is vector addition and multiplication of numbers is rotation and stretching of vectors.

In the generic space of the fully completed integration network, specific geometric or number properties are absent. All that is left is the more abstract notion of two operations on pairs of elements, such that each operation is associative, commutative, and has an identity element; each element has under each operation an inverse element; and one of the two operations is distributive with respect to the other. Something with this structure is called by mathematicians a "commutative ring." It is typically manipulated unconsciously by mathematicians who study geometry, arithmetic, or trigonometry until it becomes itself an object of conscious study in mathematics. In the development of complex numbers, it took roughly three centuries for mathematicians to reach that point.

The emergence of the concept of complex numbers with arguments and magnitudes displays all the properties of blending. There is an initial cross-space mapping of numbers to points in planar geometric space, a generic space, a projection of both inputs to the blend, with numbers fused with geometric points, emergent structure by completion (arguments and magnitudes), and by elaboration (multiplication and addition reconstrued as operations on vectors).

The blend takes on a realist interpretation within mathematics. It constitutes a new and richer way to understand numbers and space. Yet it also retains its connections to the earlier conceptions provided by the input spaces. Conceptual change of this sort is not just replacement. It is the creation of more elaborate and richly connected networks of spaces.

The evolution of the concept of complex numbers highlights the deep difference between naming and conceptualizing; adding expressions like $\sqrt{-1}$ to the domain of numbers, and calling them numbers, is not enough to make them numbers conceptually, even when they fit a consistent model. This is true of category extension in general.

Complex numbers are exotic in the history of human conception. They show up only yesterday, relative to the fifty-thousand year history since the Upper Paleolithic. They are even very recent relative to the invention of writing perhaps eight thousand years ago. How is something like this invented, and how can it stick? The answer here is that we have now achieved a blend in which the structure is made congenial by small spatial stories. Complex numbers are just vectors in two-dimensional space. To "perform operations" on them is just to manipulate objects. Adding two of them is just laying the origin of one vector down on the endpoint of the other to "reach" a new point, which is the sum of the two numbers. Multiplying them is just stretching and rotating. This is a lovely little spatial story, at human-scale. The result is a coherent conceptual packet, at human scale, involving actors as movers and manipulators of objects.

I do not mean to suggest that mathematical thinking is simple, or that the accomplished mathematician cannot achieve conceptions that go beyond simple spatial stories. But the history of mathematics and the cognitive science of mathematics suggest that even the most sophisticated and recent mathematics is indebted for its existence to our ability to blend strange mental arrays with simple spatial stories to produce human-scale little narratives that ground our thinking. In many cases, these cognitively congenial, story-based conceptions count as the accurate versions grasped and deployed by experts. In other cases, they serve as the memorable root, a kind of cognitive gnomon that serves very well, but which the experts know how to correct in those cases where they start to fail. This is the case, cognitively, in every domain, from finance to astronomy. We use Newtonian theoretical physics all the time, until we get into the ranges where we need to correct ourselves from being misguided. In mathematics, we can use the prior blends of the real number line, for example, just fine, without dealing with roots of negative numbers, but we know how to adjust once we get into that area, and activate the more sophisticated blend. We see this in pedagogy at every stage. My two older children, for example, have completed third grade in the United States and so have had pounded into their head over the course of a year the rational number blend. By now, the decimal expansion of a fraction seems perfectly natural to them But here is an interesting bit: when they learn that the decimal expansion of a number can go on indefinitely, they think about it. They understand that with each additional decimal place, the magnitude of the decimal expansion gets a little bigger. At first, they focused on this only for irrational numbers, until they realized that it's just the same for a rational number like one-third. How can it be, they ask, that the decimal expansion keeps getting bigger but the number is finite? Why does it not keep getting bigger? They wrestle with this and are on the verge of grasping it. But at the moment, they are in the blend of a small spatial story at human scale, where if you keep on going, and you go farther with every step, then the distance you travel is unbounded.

We can forgive them, partly because we remember that Zeno stumped even the brilliant ancient Presocratic philosophers with a similar situation that seemed so contrary to human conception as to count as a paradox, but also because at each stage in the history of mathematics, we see that even elite mathematicians get confounded conceptually. The greatest mathematicians of their ages wrestled with the concept of complex numbers and often failed, in whole or part, while the merest amateur now can perform operations with complex numbers that would have amazed Wallis. The difference is that we have now achieved a cognitive compression that is congenial to human understanding because it is based in small spatial stories. The future of mathematics is the achievement of ever further such blends of mathematical structures with patterns that the human mind can grasp and manipulate.

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