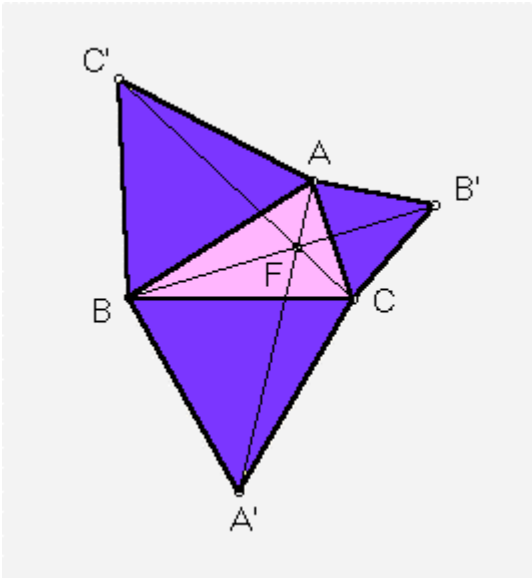
FERMAT POINT

In the plane of any triangle ABC, let

A'BC = the equilateral triangle shown on side BC,
AB'C = the equilateral triangle shown on side CA,
ABC' = the equilateral triangle shown on side AB.

The lines AA', BB', CC' meet in the *Fermat point*, said to be the first triangle center discovered after ancient Greek times. The great French mathematician Pierre Fermat, posed as a problem the search for a point P for which the sum PA + PB + PC of the distances from P to the vertices is as small as possible. Torricelli proved that the Fermat point, labeled F in the diagram, is the solution if each angle of triangle ABC is less than 120 degrees. Sometimes, F is called the Fermat-Toricelli point.



The Fermat point is also known as the *1st isogonic center*, the roots *iso* and *gon* meaning *equal-angle*. This is because the angles BFC, CFA, AFB are all equal. (The *2nd isogonic center* is obtained using the *other* three equilateral triangles on the sides of triangle ABC.

NINE-POINT CENTER

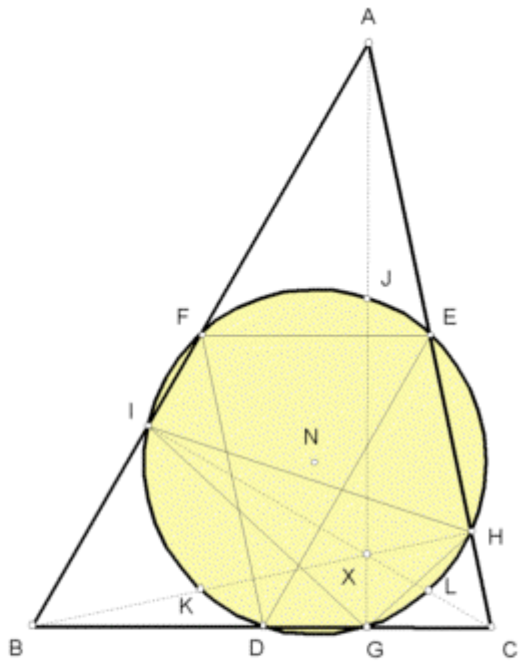
In the plane of any triangle ABC, let

D = the midpoint of side BC,
E = the midpoint of side CA,
F = the midpoint of side AB;

G = foot on BC of altitude from A,
H = foot on CA of altitude from B,
I = foot on AB of altitude from C;

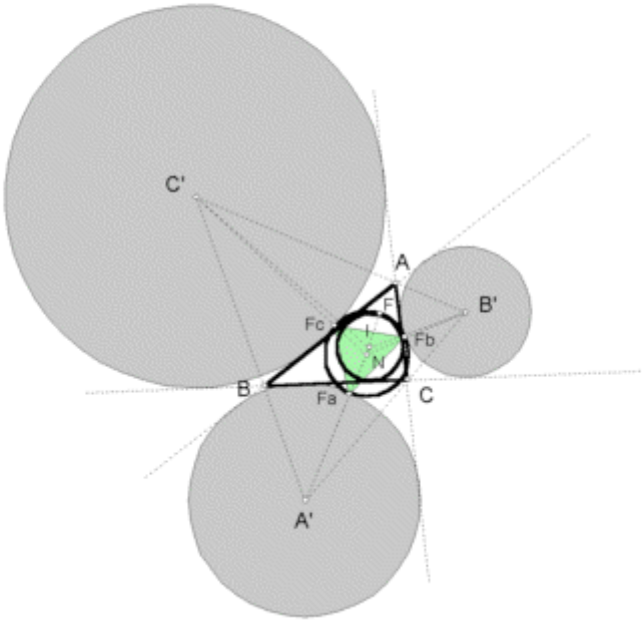
J = midpoint of segment AX,
K = midpoint of segment BX,
L = midpoint of segment CX.

(X, the orthocenter, is where the altitudes AG, BH, CI meet.)



As you see in the sketch, a circle passes through all nine of the points D,E,F,G,H,I,J,K,L. It is the *nine-point circle* of triangle ABC, and its center, N, is the *nine-point center*.

FEUERBACH POINT



Suppose ABC is a triangle. According to the very famous Feuerbach Theorem, the incircle and nine-point circle meet in a point. Labeled F in the figure, it is called the *Feuerbach point*. Trilinear coordinates for F are

1 - cos(B - C):
1 - cos(C - A):
1 - cos(A - B).

The points labeled Fa, Fb, Fc are the vertices of the *Feuerbach Triangle*. Although not shown in the figure, the lines A-to-Fa, B-to-Fb, C-to-Fc meet in the harmonic conjugate of F, having trilinears

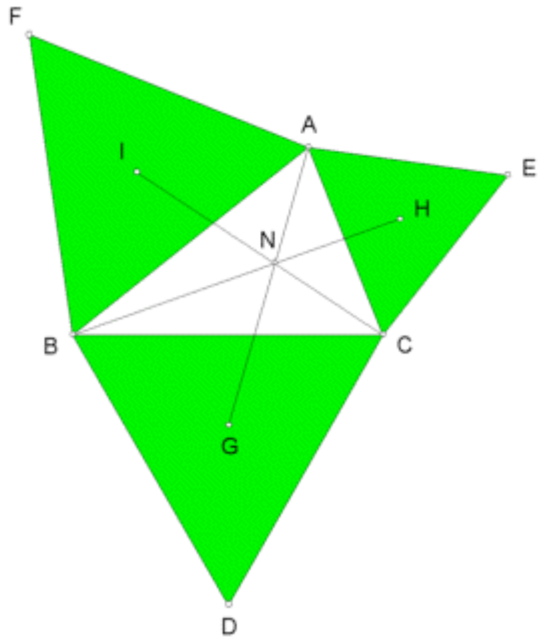
1 + cos(B - C):
1 + cos(C - A):
1 + cos(A - B)

A discussion of the Feuerbach point and biography of K. W. Feuerbach (1800-1834) are given in

Dan Pedoe, *Circles: A Mathematical View,* Mathematical Association of America, Washington, D. C., 1995.

NAPOLEON POINTS

Suppose ABC is a triangle. Let points D,E,F be points, as in the figure, for which the three triangles DBC, CAE, ABF are equilateral. Let



G = center of triangle DBC,
H = center of triangle CAE,
I = center of triangle ABF.

The lines AG, BH, CI meet in a point. Labeled N, it is called the *first Napoleon point.*

Trilinear coordinates for N are

csc(A + pi/6): csc(B + pi/6): csc(C + pi/6).

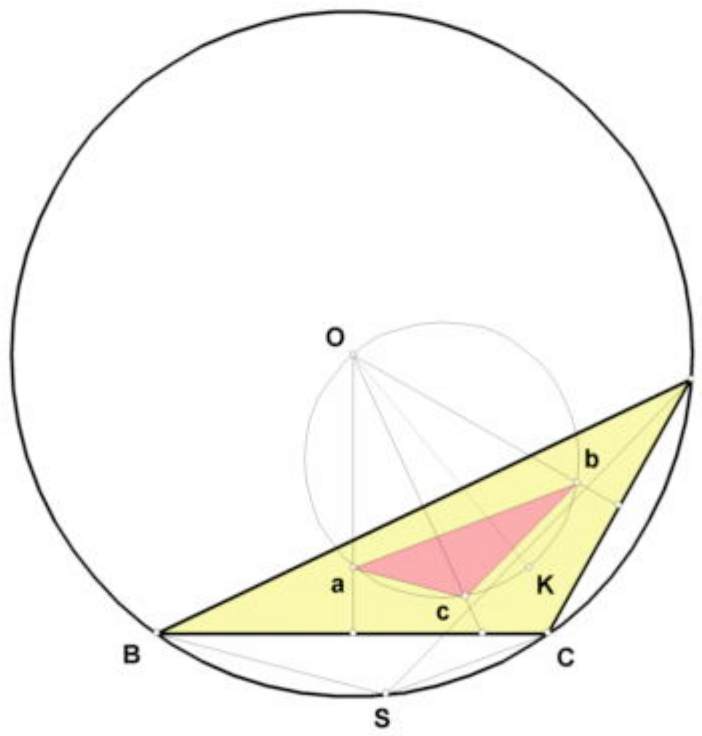
If the three equilateral triangles point inward instead of away from triangle ABC, the three lines AG, BH, CI meet in the *second Napoleon point*, with trilinears

csc(A - pi/6): csc(B - pi/6): csc(C - pi/6).

For a geometric discussion, see

John Rigby, "Napoleon revisited," *Journal of Geometry* 33 (1988) 129-146.

STEINER POINT



The Steiner point of a triangle ABC is constructed as follows: First, let O be the circumcenter and K the symmedian point of ABC. The circle having segment OK as diameter is the *Brocard circle*. The line through O perpendicular to line BC passes through the Brocard circle in another point, a; similarly, obtain points b and c. The triangle abc is the *1st Brocard triangle.* Now, construct the line through A parallel to line bc, the line through B parallel to line ca, and the line through C parallel to line ab. These three lines concur in the Steiner point, S.

Steiner described this point in a different way in 1826. Forty years later, in

J. Neuberg, "Sur le point de Steiner," *Journal de mathématiques spéciales* 1886, p. 29,

the point was constructed as described here and was given Steiner's name.

Steiner also first described the ellipse passing through the vertices A, B, C and having the centroid of triangle ABC as its center. This *Steiner ellipse* is the ellipse of least area that passes through these points. It has in common with the circumcircle of ABC not only the points A, B, C but also the Steiner point.

Homogeneous trilinear for the Steiner point are

bc/(b^2 - c^2) : ca/(c^2 - a^2) : ab/(a^2 - b^2),

where a,b,c denote the sidelengths of sides BC, CA, AB of triangle ABC. You can see easily without a pencil that the Steiner point lies on the Steiner ellipse, which has equation

1/ax + 1/by + 1/cz = 0.