FERMAT POINT

In the plane of any triangle ABC, let

A'BC = the equilateral triangle shown on side BC,
AB'C = the equilateral triangle shown on side CA,
ABC' = the equilateral triangle shown on side AB.

The lines AA', BB', CC' meet in the Fermat point, said to be the first triangle center discovered after ancient Greek times. The great French mathematician Pierre Fermat, posed as a problem the search for a point P for which the sum PA + PB + PC of the distances from P to the vertices is as small as possible. Torricelli proved that the Fermat point, labeled F in the diagram, is the solution if each angle of triangle ABC is less than 120 degrees. Sometimes, F is called the Fermat-Toricelli point.

The Fermat point is also known as the 1st isogonic center, the roots iso and gon meaning equal-angle. This is because the angles BFC, CFA, AFB are all equal. (The 2nd isogonic center is obtained using the other three equilateral triangles on the sides of triangle ABC.)
NINE-POINT CENTER

In the plane of any triangle ABC, let

D = the midpoint of side BC,
E = the midpoint of side CA,
F = the midpoint of side AB;

G = foot on BC of altitude from A,
H = foot on CA of altitude from B,
I = foot on AB of altitude from C;

J = midpoint of segment AX,
K = midpoint of segment BX,
L = midpoint of segment CX.

(X, the orthocenter, is where the altitudes AG, BH, CI meet.)

As you see in the sketch, a circle passes through all nine of the points D,E,F,G,H,I,J,K,L. It is the *nine-point circle* of triangle ABC, and its center, N, is the *nine-point center*. 
Suppose ABC is a triangle. According to the very famous Feuerbach Theorem, the incircle and nine-point circle meet in a point. Labeled F in the figure, it is called the Feuerbach point. Trilinear coordinates for F are

1 - \cos(B - C):
1 - \cos(C - A):
1 - \cos(A - B).

The points labeled Fa, Fb, Fc are the vertices of the Feuerbach Triangle. Although not shown in the figure, the lines A-to-Fa, B-to-Fb, C-to-Fc meet in the harmonic conjugate of F, having trilinears

1 + \cos(B - C):
1 + \cos(C - A):
1 + \cos(A - B)

A discussion of the Feuerbach point and biography of K. W. Feuerbach (1800-1834) are given in

Suppose ABC is a triangle. Let points D, E, F be points, as in the figure, for which the three triangles DBC, CAE, ABF are equilateral. Let

\[ G = \text{center of triangle DBC}, \]
\[ H = \text{center of triangle CAE}, \]
\[ I = \text{center of triangle ABF}. \]

The lines AG, BH, CI meet in a point. Labeled N, it is called the first Napoleon point.

Trilinear coordinates for N are

\[ \csc(A + \frac{\pi}{6}) : \csc(B + \frac{\pi}{6}) : \csc(C + \frac{\pi}{6}). \]

If the three equilateral triangles point inward instead of away from triangle ABC, the three lines AG, BH, CI meet in the second Napoleon point, with trilinears

\[ \csc(A - \frac{\pi}{6}) : \csc(B - \frac{\pi}{6}) : \csc(C - \frac{\pi}{6}). \]

For a geometric discussion, see

STEINER POINT

The Steiner point of a triangle ABC is constructed as follows: First, let O be the circumcenter and K the symmedian point of ABC. The circle having segment OK as diameter is the Brocard circle. The line through O perpendicular to line BC passes through the Brocard circle in another point, a; similarly, obtain points b and c. The triangle abc is the 1st Brocard triangle.

Now, construct the line through A parallel to line bc, the line through B parallel to line ca, and the line through C parallel to line ab. These three lines concur in the Steiner point, S.

Steiner described this point in a different way in 1826. Forty years later, in J. Neuberg, "Sur le point de Steiner," *Journal de mathématiques spéciales* 1886, p. 29,

the point was constructed as described here and was given Steiner's name.

Steiner also first described the ellipse passing through the vertices A, B, C and having the centroid of triangle ABC as its center. This Steiner ellipse is the ellipse of least area that passes through these points. It has in common with the circumcircle of ABC not only the points A, B, C but also the Steiner point.

Homogeneous trilinear for the Steiner point are

\[
bc/(b^2 - c^2) : ca/(c^2 - a^2) : ab/(a^2 - b^2),
\]

where a,b,c denote the sidelengths of sides BC, CA, AB of triangle ABC. You can see easily without a pencil that the Steiner point lies on the Steiner ellipse, which has equation

\[
1/ax + 1/by + 1/cz = 0.
\]