# Mathematics and Narrative: Natural Mathematics in the French Eighteenth Century
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Buried deep in the "Meeting Statement" for this conference lies the following paragraph:

> There is no question, of course, that logical abstraction and deductive rigor are, and will remain, the cornerstones of mathematical knowledge. But the acceptance of the logico-deductive nature of the science should not exclude the realization that one can speak intelligently, and occasionally even profoundly, about mathematics without always moving down the long-familiar tracks of the definition-axiom-proof triad.

As a historian, I am in no position to rule on what "the cornerstones of mathematical knowledge" are or will be in the twenty-first century. However, I am in a position to point out the "logical abstraction and deductive rigor" have not always been accepted as cornerstones of mathematical legitimacy, and to move into a time and place in which mathematical thinkers were resolutely resist allowing their subject be described in these terms. My goal in this is to point out the kinds of wondrous flexibilities that have sustained works we now recognize as mathematical in myriad different times and places.

The story begins in the middle of the eighteenth century in France, when the French intellectual community was in the midst of assimilating Newtonian mechanics. From a traditional mathematical point of view, this process involved juggling Newton's and Leibniz's rather different views of the calculus, neither of could claim to meet standards of "logical abstraction and deductive rigor". Leibniz's calculus was symbolically powerful; the dy/dx notation allowed problems to be approached within an algebraic framework that made conceptually difficult processes, like taking the anti-derivative, seem easy. So, for example, the derivative of the equation $\mathbf{y = x^2}$ would be $\mathbf{dy/dx = 2x}$. Taking the anti-derivative would entail simply "multiplying both sides by **dx**" to get $\mathbf{dy = 2xdx}$, then integrating both sides ($\mathbf{\int dy = \int 2xdx}$) to get $\mathbf{y = x^2 + C}$. This appeared very nice algebraically, but it was not at all clear what the process of "multiplying both sides by **dx**" meant, because considering the meaning of **dx** on its own, led directly into a world of infinitesimals that was widely recognized to be fraught with conceptual and philosophical perils.

Newtonians hoped to avoid these problems by using a fluxional notation that was firmly grounded in the physical world of the <u>Principia.</u> The price for this conceptual clarity was high, however. Newton's fluxional notation did not allow the easy manipulations of Leibniz's symbols. What was merely an inconvenience for simple problems like taking the anti-derivative of an fluxional equation like **[y with raised dot over it] = 2x** became all but impossible for more complex problems; Leibnizian symbols could cut through problems concerning functions of more than one variable in ways that the Newtonians could not conceive, for example. To make matters worse, even as they struggled to defend the importance of their conceptual high ground, Bishop Berkeley's critique of the clarity of Newtonian conceptions undermined it from within. [1]

In the France of the 1750s, however, there was little sense that mathematics was faced with a crisis of rigor. Analysis was a wonderfully rich and developing field, whose powers were understood in myriad different ways. Some form of the ideal of "logical abstraction and deductive rigor" may be found among them, but not as universally recognized values. In fact, for the enlightened French, the very word "rigor *[rigueur]*" was often disparaged as too rigid and restricted to capture the riches of mathematical development. One alternative approach was through a progressive historical narrative that did not have to choose between the meanings of geometry and the relations of algebra but rather encompassed them both. In the second half of the century histories of mathematics were used not only as ways of speaking "intelligently and occasionally even profoundly, about mathematics" but also as ways to ground it, and give it legitimacy. It was only after the French Revolution of 1789 radically fundamentally altered both the institutional context of mathematics and the progressive understanding of its history that the supports for this tradition slowly gave way that a mathematics grounded in "logical abstraction and deductive rigor" could rise to take its place.

## I. Mathematics and Natural Knowledge

[1] The classic treatment of this story is to be found in Carl Boyer, The History of the Calculus and its Conceptual Development (Concepts of the Calculus) With a forward by Richard Courant (New York; Dover, 1959). For a careful look at the symbolic strengths of the Leibnizian approach, see H. J. M. Bos, "Differentials, higher order differentials and the derivative in the Leibnizian calculus," Archive for History of Exact Sciences 14 (1974): 1-90.

The last years of the 1740s in France saw the publication of a crowd of works that together signaled a turning point in intellectual life. Montesquieu's Esprit de lois was published in 1748, Condillac's Traité des systemes, Diderot's Lettres sur le aveugles, Buffon's *Histoire naturelle*, all appeared in 1749, Rousseau's Discours sur les sciences et les arts was published 1750. The crowning touch came just a year later, a group that characterized itself as "a society of men of letters," published the first three volumes of the *Encyclopédie*. [2]

The Encyclopédie is often touted as among the first modern encyclopedias, but there is an exuberance to the work that couldn't be further from the staid and weighty tomes of works like the Encyclopedia Brittanica. A wide range of people contributed to the seventeen volumes of the *Encyclopédie* that appeared between 1751 and 1772, but the major power behind the enterprise was the editor, Denis Diderot, who followed his ideas and passions through article after article. The whole was alphabetically organized but behind this staid exterior Diderot constructed a subversive network of cross-references; readers who made it to the end of orthodox articles might find themselves directed to a thicket of anti-establishment discussion in articles like "Certainty", "Chance", or "Doubt".

Also embedded in the Encyclopédie is a series of mathematical articles on subjects like "Arithmetic", "Analysis", "Derivative" that taken together form a comprehensive text of eighteenth-century mathematics. The author of these mathematical

[2] *Encyclopédie, ou DictionnaireRaisonné des Sciences des Arts et des Métiers.* 17 vols. (Paris, 1751-65). For a comprehensive treatment of this work see: John Lough, *The Encyclopédie* (New York: D. McKay Co., 1971). All English-language quotations from the "*Discours Préliminaire"* to these volumes are from: Jean Le Rond D'Alembert, Preliminary Discourse to the Encyclopedia of Diderot. Translated by Richard N. Schwab with the collaboration of Walter E. Rex with an

articles was Jean Le Rond D'Alembert, who at its inception was co-editor of the *Encyclopédie*. D'Alembert had first made a name for himself with "D'Alembert's principle" published in a *Traité de dynamique* of 1743. A year later he published his *Traité de l'equilibre et du mouvement des fluide*, from which he moved on to a lifetime of mathematical creativity.

D'Alembert's position as mathematician among the *encyclopédists* was a somewhat difficult one. In the article "*Encyclopédie*" that appeared in the fifth volume, Diderot trumpeted "a general movement [away from mathematics and] towards natural history, anatomy, chemistry and experimental physics."[3] Many people, including not only Diderot, but also the writer Voltaire, and the natural historian Georges-Louis Le Clerc, Compte de Buffon, had all begun their careers with mathematical interests but over the course of the 1740s and 1750s they had all come to the conclusion that the study of mathematics was not a valuable way to generate insight or understanding.[4] In this group D'Alembert might be seen as a rock standing firm against the shifting tides of mathematical fortune, but he was not unaffected by the ideas that swirled about him. He defended mathematics, but the mathematics he advocated and pursued was defined and developed within the value system of an enlightened *philosophe*.

D'Alembertian enlightened mathematics might be described by a variety of positive characteristics, but the focus of this paper is a negative one; enlightened mathematics was not rigorous. This lack of rigor was not just a response to the result of

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introduction and notes by Richard N. Schwab. (Indianapolis: Bobbs-Merrill Educational Publishing Company, Inc., 1963), this one from page 8.

[3] *Encyclopédie,* sv. *"Encyclopédie"* quoted in Thomas L Hankins, Jean d'Alembert: Science and the Enlightenment (New York: Gordon and Breach, 1970) 99.

[4] Ibid.

philosophical difficulties in the calculus, nor the still-split legacy of the seventeenth century; enlightened mathematics was not rigorous because rigor—whether geometrical or algebraic—was a negative value in the mid-eighteenth century world of the Encyclopédie. To have standing in this world, mathematics had to be rooted in more organic soil. The situation was clearly laid out in the introductory essay to Buffon's Histoire naturelle: "*Premier Discours: De la maniere d'étudier et de traiter l'histoire naturelle*". In this essay Buffon set out the epistemological foundation for the sprawling thirty-seven volumes of observational natural history that his work was eventually to become. Buffon's focus was natural historical, but his vision was arguably equally important for eighteenth-century mathematics.

Buffon's enormous Histoire naturelle was conceived in opposition to the artificial economy of Linnaeus's classificatory approach. In answer to Linnaeus's determined focus on the reproductive parts of plants, Buffon defended a comprehensive observational approach in which the key to success lay in knowing "how to distinguish what is real in a subject from what we arbitrarily put there." If scientific investigators were to focus on the real at all times, Buffon claimed, "Disputes would cease and all would unite to advance along the same path following experience. Finally, we would arrive at the knowledge of all the truths which are within the competence of the human mind." [5]

Strong emotional and moral overtones undergirded Buffon's insistence on open-minded observation: "Nature's mechanism, art, resources, even its confusion, fill us with admiration. Dwarfed before that immensity, overwhelmed by the number of wonders, the

[5] All English-language quotations from Buffon's "*Premier Discours*" are taken from John Lyon, and Sloan, Philip R, From Natural History to the History of Nature (Notre Dame: Notre Dame University Press, 1981), this one from page 127.

human mind staggers. . . . What an impression of power this spectacle offers us! What sentiments of respect this view of the universe inspires in us for its Author!" [6] "And the initial thought that follows is humbling self-reflection." [7] This humble self-reflection is clearly a critical part of the value of natural history for Buffon: he recommends that the subject be taught to adolescents "at that age when they might begin to think that they already know quite a bit. Nothing is more apt to lessen their conceit and make them feel how much there is that they are ignorant of. . . ." [8] The study of natural history is valuable because it draws us outside of ourselves, and confronts us with a reality that is larger than we can ever be.

Buffon only turned briefly to mathematics and mathematical physics in the last couple of pages of his "*Premier Discours*." He knew the subject well. His first published work was a 1740 translation into French of Newton's Method of Fluxions and Infinite Series, and just one year before he published his "*Premier Discours*" he had been an active participant with d'Alembert, Leonhardt Euler and Alexis Clairaut in a dispute about the mathematical form of Newton's theory of gravitation. Whatever his mathematical proficiency, however, the view of mathematics Buffon developed in the "*Premier Discours*" was highly negative. In a scientific enterprise valued because it confronts us with a reality fundamentally greater than we, mathematics is empty and solipsistic; it is "reduced to the identity of ideas, and has nothing of the real about it."[9] For Buffon mathematics was so empty that it could not even serve as a helpmate to more factually grounded efforts. He could not deny the power of Newton's cosmology but then

[6] Ibid., 101.
[7] Ibid., 98.
[8] Ibid., 99-100.

warned, "there are very few subjects in physics in which the abstract sciences can be applied so advantageously." [10] Buffon even cautioned against following Newton's mathematical lead in developing mechanics.

> The true goal of experimental physics is . . . to experiment with all things that we are not able to measure by mathematics, all the effects of which we do not yet know the causes, and all properties whose circumstances we do not know. That alone can lead us to new discoveries, whereas the demonstration of mathematical effects will never show us anything except what we already know. [11]

Diderot applauded Buffon's attack on empty mathematical knowledge: "One of the truths which have been announced recently with the greatest courage and force . . .is that the region of the mathematicians is an intellectual world where what are assumed to be rigorous truths lose this advantage completely when carried to our earth."[12] His mathematical co-editor however, fought back. In the first of the three volumes of the Encycyopédie D'Alembert countered Buffon's "*Premier Discours*" with a "*Discours Préliminaire*". Whereas Buffon had attacked mathematics as meaningless and hence empty, D'Alembert defended it as essentially grounded in the empirical world. Whereas Buffon saw mathematical objects to be the product of the human mind, D'Alembert saw them as obtained from the external world by a process of purifying abstraction. D'Alembert argued that mathematics was not empty; geometrical validity rested forms at least as real as the physical objects from which they had been abstracted; algebra too arises from the physical world and algebraic results return there. The values of natural knowledge that Buffon had so clearly articulated—the unanimity, awe and humility—

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[9] Ibid., 123.
[10] Ibid., 126.
[11] Ibid.
[12] Quoted in Thomas L Hankins, Jean d'Alembert(New York: Gordon and Breach, 1970) 89.

could all be reached through mathematical study because mathematical objects were natural and therefore legitimate.

D'Alembert's description of mathematics in the "*Discours Préliminaire*" was just a first step towards answering those like Buffon and Diderot, who found the subject completely and uselessly empty. He continued to develop his views in the article "*Elémens des sciences*" that appeared in the 5th volume of the *Encyclopédie*. The title of this six-page double-column article is a clear reference to Euclid's Elements, but d'Alembert's goal of grounding mathematics naturally could never be achieved by "a more artificial method, like that Euclid followed in his '*Elémens'."* He railed against the emptiness of Euclid's "axioms," which he wanted to replace with natural "principles." To d'Alembert, "the difficulty with which he [Euclid] proceeds makes it easily apparent that this kind of precarious and forced rigor can never be anything but improper;"[13] in his article the Frenchman was proposing an *élémens* that would displace the Euclidean model forever.

D'Alembert's dismissal of Euclidean rigor was something of a commonplace among enlightened mathematicians. In 1841, Alexis de Clairaut had written in the preface to his *Élémens de Géométrie:* "[Euclid's] geometry had to convince stubborn sophists who prided themselves on refusing [to believe] the most evident truths; it was necessary then that geometry have the help of forms of reasoning to shut the idiots up. But times have changed. All reasoning which applied to that which good sense knows in advance is a pure loss and serves only to obscure truth and disgust the reader." [14] In a similar vein

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[13] *Encyclopédie,* sv. *Elémens des sciences.* (Unless otherwise noted or cited, all translations from the French are mine.)
[14] A.C. Clairaut, *Elémens de Géométrie*. (Paris, 1741) p. 4.

LaChapelle tried to distance his *Institutions de Géométrie* from "the tortuous discussion of punctilious metaphysicians who insist that geometry have its articles of faith like theology."[15]

In his "*Elémens des sciences"* d'Alembert went beyond criticizing Euclid, however, and tried to create an alternative. He defined the elements of any subject as "the primitive and original parts of which the whole is formed." He then described the ideal form of an elements.

> Let us suppose that this science be entirely treated in a work in such a way as to range before us the whole sequence of propositions, as much general as particular, which form the whole, and that these propositions are presented in the order which is the most natural and the most rigorous.[16]

The "chains of reasoning" that tied together the web of knowledge were mirrors of real chains of relation that tied together the natural world. When an "*élémens"* was complete, "the human mind [*l'esprit humain*], participating then in the supreme intelligence, would see all their knowledge as reunited under an indivisible point of view." [17]

D'Alembert recognized that his program to naturalize mathematics required him to clarify what he meant by rigor. "One asks which of the two qualities, simplicity [*facilité*] or exact rigor [*la rigueur exacte*] should be preferred in an elements." To which he replied: "This question assumes something false; it assumes that exact rigor can exist without simplicity [*facilité*] and that is false: the more rigorous a deduction the easier it is to understand because rigor consists in reducing everything to the simplest *[plus facile]* principles. From which it follows that rigor, properly understood, necessarily entails the

[15] M. de LaChapelle, *Institutions de Géométrie enrichies de notes critiques et philosophique sur la nature et les developpements de l'esprit humain.* (Paris, 1757)
[16] *Encyclopédie, s.v. "Elémens des sciences."*
[17] Ibid.

most natural and direct method. "The more the principles are arranged in the appropriate order [*l'ordre convenable*] the more the deduction will be rigorous."[18] This re-definition of rigor can be taken as d'Alembert's answer to Buffon's charge that mathematics was artificial and empty; it opened for him a mathematics as significant and meaningful as Buffon's natural history.

The next step in D'Alembert's program to defend mathematics was to claim for his subject the kinds of virtues Buffon has reserved for students of natural history. Just two volumes after "*Elémens des sciences*", in the seventh volume of the *Encyclopédie*, d'Alembert published "*Géometre"* in which he argued that mathematics should be the heart of an enlightened education. Learning mathematics would educate the whole person, "subtly preparing the way for the philosophical spirit [*l'esprit philosophique*]"; in the large, a mathematically educated populace would be an enlightened populace, "disposing an entire nation to receive the light which this spirit can pour forth." [19]

The specific traits by which this transformation of the *l'esprit humain* was to be accomplished were multifarious. D'Alembert pointed to "*la justesse de l'esprit* to seize reasonings and untie paralogisms, *facilité de la conception* to understand promptly, *l'étendue* to comprehend all at once the different parts of a complicated demonstration, *la memoire* to retain the principle propositions." In addition, the best *géometre* possesses "other qualities still less common: depth, invention, power and wisdom." In short, when pursuing mathematics the *géometre* is not only cultivating "*l'esprit géometre"* but, by extension, *"l'esprit géometrique,* that is to say *l'esprit de methode & de justesse."* [20]

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[18] Ibid.
[19] *Encyclopédie,* s.v. "*Géometre*".
[20] Ibid.

D'Alembert's efforts to define and defend an enlightened mathematics in the Encyclopédie were here cut short. His fiercely anti-clerical article "*Genève*" appeared in the same seventh-volume as *"Géometre,"* and called the full wrath of the censors down on the entire venture. D'Alembert was frightened and withdrew as editor, leaving Diderot to soldier on alone. "The reign of mathematics is over,"[21] an irritated Diderot wrote to Voltaire, and in the ten succeeding volumes he did what he could to undercut the claims of d'Alembert's "*l'esprit géometre*". His tenth-volume "*Philosophe*" evinced *l'esprit d'observation & de justesse* , as opposed to d'Alembert's "G*éometre"'s l'esprit de methode & de justesse*. At least equally pointed is the barb hidden in definition of rigor of the XIVth volume: "*Rigeur:* Severe and inflexible conformity to some given law. . .Genius has no need for *rigor* . . .The demonstrations of the mathematician [*géometre*] are *rigorous*."[22]

## II. The History of Mathematics

Diderot may have refused to credit d'Alembert's attempts to cast mathematics as an enlightened study, but others were sympathetic to d'Alembert's approach. In 1758, Jean Étienne Montucla published a two volume Histoire des Mathématiques, which positively dripped the broad humanistic vision of d'Alembert's *l'esprit géometrique*. "One of the spectacles most worth to interest a philosophical eye, is without doubt that of the development of the human spirit and the different branches of human knowledge,"[23]

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[21] Hankins, 100.
[22] Encyclopédie, s.v. "Rigueur".
[23] Jean-Etienne Montucla, Histoire des mathématiques, (Paris: 1758), vol. 1, iii.

Montucla declared, and bolstered his position with a now-lost letter from Montmort: "It seems to me that [a history of mathematics] well done, could be looked upon as the history of the human mind [*l'esprit humain*], since it is in this science more than in all others that man makes known the excellence of the gift of intelligence which God has given him to raise him above all other creatures."[24] For Montucla, as for Montmort, a properly written history of mathematics could serve as a mirror for the development of the human mind.

The underlying concept that shaped Montucla's narrative of mathematics from the ancient Greeks to the seventeenth century was progress; when properly told, the history of mathematics was a linearly developing story of ever increasing understanding. "Of all the sciences, mathematics is the one of which the path in the search for truth has been the most assured and the best sustained. . . .Their development has never been interrupted by shameful failures of which all the other parts of our knowledge offer so many humiliating examples."[25] Montucla's idea of progress was the "point of view" that elevated his history to the level of a d'Alembertian *"élémens";* it allowed him to put his facts "in order" in such a way that people could see "the relation among them *[la liaison entr'elles]*". [26] The goal of Montucla's history went beyond allowing readers to revel vicariously in the glory of past discoveries, or showing them "an easy and agreeable path" to the present, however. By showing his readers "paths taken", sources used and discoveries made, Montucla was moving them to a privileged point of view. He was

[24] This quotation is from a now lost letter from Montmort to Bernoulli that Montucla claimed as his inspiration: Ibid., viii.
[25] Ibid., xxv.
[26] Ibid., iv.

organizing the past in such a way as to enable them also "in a manner of speaking, to put their forces in order so they could more easily move ahead." [27]

The conviction that progressive history provided a uniquely productive way to structure mathematical understanding was not confined to Montucla's *Histoire.* Its mathematical impact can be seen in the early work of Joseph Louis Lagrange. In 1766 d'Alembert recommended the young Italian as a worthy successor to Euler in Frederick the Great's Berlin academy. Frederick was not particularly interested in mathematics, which he complained, "dries up the mind", [28] but he was an admirer of d'Alembert. D'Alembert had also been critical of Euler, whom he described as "a modern mathematician, who lives in Germany as a *philosophe*" but nonetheless insisted on pursuing even the most elementary mathematics with "such obscure reasoning that the reader can only tend to doubt." [29]

If Frederick had hoped that Lagrange's mathematics would be less dry than Euler's he was perhaps disappointed. Over the course of his twenty years in Berlin, Lagrange radically changed the face of eighteenth century mechanics by bringing together the sprawling field with the symbolical power of the new analysis. "No figures will be found in this work," he crowed in the introduction to his *Mechanique Analitic*. "The methods I present require neither constructions nor geometrical or mechanical arguments, but solely algebraic operations subject to a regular and uniform procedure."[30] From d'Alembert's or Montucla's point of view, however, Lagrange's *Mechanique Analitic* can

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[27] Ibid., v.
[28] Cajori, (1927):122.
[29] *Encyclopédie*, s.v. "*Elémens des sciences*." (Translation mine).
[30] J. L. Lagrange, Analytical Mechanics. Trans and ed by Auguste Boissonnade and Victor N Vagliente. (Boston: Kluwer, 1997) p. 7.

be seen as a triumph, because mathematical history took geometry's place as the ground for Lagrange's mechanics. Echoing the language of the *"Elémens des sciences"* d'Alembert's protégé declared his intention to present "the various principles. . . . from a single point of view," [31] which he established by introducing each section of his work with a history.

After the French revolution ushered in a new age, an elderly Montucla decided to expand and re-issue his *Histoire des Mathématiques.* In his first edition he had stopped with the seventeenth century, but this time he intended to bring the story to the present, by including the eighteenth century as well. It was a mammoth task, however, and the old man moaned under its weight.

> How many books to read, to summarize, to compare, to bring together the material of my edifice! I will add to that the knowledge of the principal languages of Europe, in order to be able to consult a crowd of non-translated books. I say nothing of the necessity of bringing to these researches an adequately profound knowledge of all the parts of mathematics of which the system is so vast.[32]

In the event, death freed him from the effort and the last two volumes of his second edition were primarily written by others. Most notable among these helpers was Sylvestre François Lacroix, who wrote the section on the development of analysis in the eighteenth century. [33]

Coverage was the major goal of the treatment of eighteenth century analysis Lacroix wrote for Montucla. The material is sometimes organized around subjects like the geometry of curved surfaces, series, differentials, logarithms, probability; sometimes

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[31] Ibid.
[32] Jean-Etienne Montucla, *Histoire des mathématiques, Nouv. ed. considérablement augm., et prolongée jusqua vers l'époque actuelle* (Paris: H. Agasse, [1799]-1802) I: iv.

around methods like the method of inverse tangents, Newtonian fluxions, methods of elimination; sometimes around historical peculiarities like the Newton-Leibniz controversy, the treatment of differentials in one variable by Leibniz, the Bernoullis, Cotes, deMoivre, d'Alembert etc. This lack of commitment is particularly striking in LaCroix's treatment of the calculus which includes both of the major foundational alternatives—Newtonian fluxions and Leibnizian infinitesimals—without making any attempt to compare them directly or make a choice between them. He tried to eliminate nothing because all the "paths taken" could be relevant to rallying mathematical powers and enabling them to move more easily forward.

There is one jarring exception to Lacroix's relaxed ecumenical approach, however. This is the chapter he devoted to two major debates: one sparked by the challenge of Berkeley's Analyst, the one among Rolle, Varignon, Saurin et.al. Both were debates over rigor and LaCroix had no patience for either of them. He describes Maclaurin's attempt to establish the rigor of the calculus as "of a prodigious length demanding a contention of the spirit of which I think few mathematicians are capable today." [34] Of Rolle, who challenged the rigor of Leibniz's calculus, he fumed: "It is never excusable to be wrong in geometry, and to oppose oneself by passion and jealousy to discoveries proper to accelerating the progress of the sciences."[35] In his conclusion, Lacroix dismissed both controversies in disgust.

> After works so solid and responses so victorious to all the difficulties raised against the method of fluxions or the differential and integral calculus, there could

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[33] For Lacroix see: René Taton, "Sylvestre-Francois Lacroix: Mathématicien, Professeur et Historien des Sciences." *Actes du VIIe Contrès International d'Historie des Sciences.* (Jerusalem, 1953) 588-93.
[34] Montucla ([1799]-1802) 3: 118-119.
[35] Ibid., 116.

> not be any but some ignorant soul or false spirit of a most rare kind, who could raise the question [of rigor] again. One must expect that from time to time one will see attacks, since every day the most simple truths are open to question; but one does not have to pay any attention unless the people who offer them give some proof of their understanding. . . . The life of a man who cultivates the sciences would be a perpetual quarrel.[36]

Lacroix's attack on early eighteenth century discussions of rigor points to the one constant that underlies the catholic acceptance of his otherwise loose historical narrative; the history of mathematics is progressive. The dynamic he describes may seem chaotic and different results contradictory, but all was moving forward toward a common goal of increasing understanding. There was nothing to be gained from insisting that developing mathematics adhere to strict standards of "logical abstraction and deductive rigor". On the contrary such concerns were liable to stop mathematics—and with it the development of *l'esprit humain*—in its tracks.

III. History and mathematics after the revolution

Lacroix was at least as much an educator as he was a historian; at one time or another he taught mathematics at virtually all of the post-revolutionary Parisian educational institutions—the *Ecole Normale*, the *Ecole Polytechnique*, the *Faculté des Sciences*, the *Collège de France*—and his *Essais sur L'Enseignement* of 1805 was a classic. He also wrote a number of textbooks of which his *Traité du Calcul* is the most well known. Judy Grabiner has explored the ways that these educational institutions impacted the way mathematics developed after the revolution; [37] certainly the demands on

[36] Ibid., 119.
[37] Judith Grabiner, The Origins of Cauchy's Rigorous Calculus (Cambridge: MIT Press, 1981).

mathematics within this educational environment significantly shaped the subject Lacroix presented in his textbooks. It did not, however, lead him to abandon the breadth of vision that characterized his historical work. The two editions of his _Traité_—the first in 1797 the second in 1810—show both the continuing importance of natural historical mathematics after the revolution and the forces at work to changed it.

These forces were at least two-fold. On the one hand, the historical rupture of the Revolution was raising serious questions about the validity of the continuous progressive view of history Lacroix had inherited from the eighteenth century. On the other, the need efficiently to teach mathematics in the post-revolutionary educational system challenged his rambling historical approach. Both historical and educational forces played a part in leading Lacroix, and others like him, to make adjustments in their views of mathematical development. However, neither was enough to force the abandonment of d'Alembert's vision of a single point of view large enough to embrace all of the complexities of eighteenth century analysis.

When Lacroix first broached the idea of writing a textbook to Pierre Simon LaPlace in 1792, the older man supported the project in terms strikingly close to those that supported Montucla's _Histoire._

> Bringing together the methods [*Le rapprochement des Méthodes*] as you intend to serves mutually to clarify them, and that which they have in common will usually be their true foundation [*vraie métaphysique*], which is why that foundation is almost always the last thing that one discovers. The genius comes as if by instinct to results; it is only in reflecting on the route which he and others have followed that it begins to generalize the methods and discover the foundations.[38]

[38] S. F. Lacroix, _Traité du Calcul differential et du calcul integral,_ Seconde edition, revue et augmentée (Paris: Chez Courcier, 1810) 1: xix.

That Lacroix quoted this in "Preface" to his work is evidence of his basic allegiance to this approach. However, his *Traité* makes no claims to be a history book. When presenting ideas in his histories, Lacroix had conscientiously followed the twists and turns of actual developments, but in his *Traité* he firmly imposed an interpretation on that variety.

Lacroix's divergence from history could be construed as a challenge to enlightened historical mathematics, and he knew it. In the "Preface" to his 1797 *Traité,* he expressly tried to explain his approach in terms of his particular position as an historical actor.

> One notices in fact in the history of mathematics, certain epochs where although the truth of particular propositions has not been altered, their systematic relationships change because of their relationships to new discoveries which have taken place. The principles have become more fecund, the details less necessary, and the generality of the methods allows one to embrace the science as a whole, despite the immense strides she has made.[39]

Lacroix's consciousness of distinct epochs in mathematical development sharply divides him from the world of Montucla's *Histoire*. The change was much greater than these two men, however. All around them was evidence that the French revolution represented a massive break in a history. Accepting the reality of such violent historical breaks meant that it was no longer obvious that even mathematics could assume a historical development that was either linearly progressive or a simple mirror of psychological development. In the years following the Revolution this kind of thinking gradually undercut the view of continuous progress that grounded enlightened historical mathematics; it became ever less obvious that the shape of past developments held the key to understanding the present.

There was also a strong pedagogical component to the weakening of Lacroix's historical approach to mathematics. By 1805, when he published his *Essais sur l'Enseignement*, Lacroix was still paying lip-service to historical mathematics but only on an elementary level. On a more advanced one "after they have imbibed the fundamental truths of the science," students were ready for an approach that "put the propositions in order so as to make their rational interconnections evident." This order would not be historical; the attempt "to educate students in the ways of their ancestors, even the most celebrated of them, must cease because the science has entered a new age which has completely changed the connections of the propositions and often their language."[40] Lagrange, who since the revolution had also been teaching in Paris, had reached essentially the same conclusion by 1806 when he wrote: ". . .in the current state of analysis we may regard these discussions [of past mathematics] as useless, for they concern forgotten methods, which have given way to others more simple and more general."[41]

These new attitudes towards both history and mathematics opened the door for Lacroix to organize his mathematical material in new ways. In the first, 1797 edition of his *Traité,* he adopted an algebraic interpretation of the derivative that Lagrange had espoused in his *Théorie des Fonctions Analytique.* In this book, also published in 1797, Lagrange can be seen as following the algebraic lead of his *Mechanique Analitic* into the calculus. His stated goal was to write a work that would separate the differential calculus

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[39] Ibid., ii.
[40] Lacroix, *Essais sur l'Enseignement en General et sur celuis des mathématiques en particulier,* 2nd ed. (Paris, 1816) 174-82.
[41] Quoted in Craig Fraser, "J. L. Lagrange's changing approach to the foundations of the calculus of variations," Archive for History of the Exact Sciences 32(1985): 152.

from all "metaphysical" considerations like those of the infinitely small or of limits. [42] Lagrange based his presentation on the well-known Taylor series expansion of a function. Given a function, Lagrange asserted, one has a host of uniquely determined derived functions provided by the Taylor series.

$$f(a+h) = f(a) + f'(a)h + f''(a)h^2/2! + f'''(a)h^3/3! \ldots$$

Lagrange claimed that this series allowed one to define derivatives of any order independently of infinitesimal or conceptual arguments. He simply defined the first derivative as the coefficient of the second term, the second derivative as the coefficient of the third term etc. In 1797, this argument convinced Lacroix, and he adopted it for the first edition of his *Traité.* Lacroix did not remain impressed with Lagrange's algebraic approach, however, and the second, 1810 edition of his *Traité* rests on a notion of the limit.

It is not difficult to see the problem that led Lacroix to abandon Lagrange's approach in his second edition. In his 1797 work, Lagrange had defined a function as "any expression of the calculus in which quantities enter in whatever manner, while the variables of the function can be assigned all possible values."[43] This definition was much broader than the definition Taylor had used when he developed his series. Therefore it led immediately to the question of whether a Taylor series could be generated for all functions that fit the broader definition. This set Lagrange and his compatriots the problem of proving that a Taylor series could be generated from all functions that fit his

[42] J. L. Lagrange, Théorie des functions analytiques (Paris: Imprimerie de la République, v (1797)) 7.
[43] Ibid., 2.

definition, and in the decade that followed the publication of his *Theorie*, neither Lagrange nor any of his compatriots was able to generate such a proof.

For Lacroix, whose goal was allow his readers to assimilate all of the wealth of eighteenth-century calculus, this posed a serious problem. The failure to prove that all Lagrange functions had Taylor series expansions meant that adopting Lagrange's approach meant you had to consider a variety of specific cases that were exceptional in one way or another. In Lacroix's view the constant need to investigate individual cases "obscure[d] the foundations of the theory *[jette sur les fondemens de la théorie, des nuages]* in a way that would not happen if one kept the trace of induction by which one arrived at the general statement *[la trace del'induction par laquelle on est arrivé à l'énonce général.]*."[44] In the second edition of his *Traité* he abandoned Lagrange's algebraic definition of the derivative in favor of a more wide-ranging definition based on an idea of the limit.

Even as Lacroix thus kept his readers open to the full wealth of eighteenth-century mathematics, he also recognized that he was molding the calculus bequeathed to him by history. "Whatever route one chooses can lead to important discoveries," he admitted, "and each point of view from which one looks at the passage from algebra to differential calculus, gives this calculus forms which, at the very least, give particular facility in the solution of certain problems." Of the various possibilities this situation left him, Lacroix chose the limit because it allowed him "to conciliate rapidity of exposition with exactitude in language, . . . allows approaching without difficulty the metaphysics of

[44] Lacroix (1810), I: xxi.

Leibnitz or the theory of the development of functions proposed by Lagrange.” [45] From a modern perspective, Lacroix’s adjustment can be seen as a move towards rigor, but for Lacroix the crucial issue was scope; as he saw it, the limit was broad enough and flexible enough that it was sufficient to organize and present all of the rich variety of his predecessor’s results.

Small differences in language can be seen separating Lacroix’s vision from d’Alembert’s. In his “*Élémens des sciences”* d’Alembert had searched for “a single point of view” from which the relations of all knowledge would come clear; forty years later Lacroix was attempting to impart a “uniform tint *[une tente uniform]*” to the material he presented. The result of d’Alembert’s mathematics would be a mathematics that was “easy to understand *[facile à entendre]*” [46], whereas Lacroix’s efforts would create “precision and clarity.” [47] In these points of detail, Lacroix may be seen as accepting a more active role in organizing his materials in order to create a more efficient mathematics. Nonetheless, even as his language leaned towards the rigor that was to come, Lacroix remained always true to the organic freedom of eighteenth century mathematics. He moved to the limit because with it he had found an approach that would encompass all of history’s riches.

V Cauchy and the end of enlightened mathematics

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[45] Ibid., xxiv.
[46] *Encyclopédie*, s.v. “*Elémens des sciences*.”
[47] Lacroix, ( 1810) I:ii.

Twelve years after Lacroix published the second edition of his *Traité* France had entered yet another epoch; Napoleon had been deposed and the Bourbons were again in power. The reshuffling of positions such a change entailed took place not only in the political world but in the institutions of science as well. The École Polytechnique was closed for several months, and when it reopened the quintessential enlightenment mathematician, Gaspard Monge no longer had a position. His replacement was the brilliant young ultra-conservative Augustin Cauchy. [48]

In 1821 Cauchy published the *Cours d'Analyse*, in which he insisted on the independence of mathematics not only from its history, but from all other areas of knowledge. "Let us then admit that there are truths other than those of algebra, realities other than those of sensible objects. Let us ardently pursue mathematics without trying to extend it beyond its domain; and let us not imagine that we can attack history with formulas nor that we can offer as moral training the theorems of algebra or of integral calculus." [49] In exchange for this enlightened breadth, Cauchy offered a calculus that could be developed with "all the rigor of geometry." [50]

Despite Cauchy's invocation of geometrical rigor, in his *Cours* he did not reach after the clarity of conception that had for so long marked geometrical understanding. What Cauchy was trying to do was to graft geometry's "deductive rigor" onto some kind of abstraction similar to that which had earlier led Lagrange to his general notion of a function. Cauchy's rigorous ideal can be seen as requiring "logical abstraction" if this

[48] For Cauchy see: Bruno Belhoste, Cauchy: Un Mathématicien Légitimiste au XIXe Siècle (Paris: Belin, 1985)
[49] Augustin-Louis Cauchy, Cours d'Analyse de L'École Royale Polytechnique (Paris: De L'Imprimerie Royale, 1821) vii.
[50] ". . .toute la rigueur qu'on exige en géométrie"

phrase is taken to mean abstractions defined in such a way they could support a deductive structure. This is arguably what Cauchy did by developing a more precise definition of the limit, which within just a few years became the basis for delta-epsilon proofs.

Eighteenth-century mathematics was significantly reduced by Cauchy's rigor; definitions of continuity cast doubt on the validity of many functions, divergent series were ruled out entirely, and imaginary numbers had to be radically reinterpreted. Cauchy explicitly recognized the limiting aspects of his approach, but in his view, they were more than compensated by his new ideal of "logical abstraction and deductive rigor". "In determining these conditions and these values, and fixing in a precise manner the meaning of the notations, I will make all uncertainty disappear," he wrote.[51]

In 1749, as he was rejecting mathematics and turning his attention to the wonders of natural history, Buffon had written:

> Since we are the creators of this sort of knowledge [mathematics], and since it takes under consideration absolutely nothing except what we ourselves have already imagined, it is impossible to have therein either obscurities or paradoxes which may be actual or impossible of resolution. A solution will always be found for these apparent difficulties through a careful examination of the premises, and by following all the steps which have been taken to arrive at the solution.[52]

This statement is striking because Buffon is here setting forward a view of mathematics very close to that Cauchy defended, almost eighty years later. Both men insisted on the separation of mathematics from other subjects. Both recognized that it was the prerogative of the mathematician to define and limit his subject according to the needs of reason. The difference between the two men is in the value that they attach to the characteristics that they agreed mathematics displays. For Buffon, they were the marks of complete irrelevance, whereas for Cauchy, they promised sanctuary from the complexities of a too-true world.

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[51] Ibid., iii.
[52] Lyon and Sloan, 124.