Σχετικά με το πλήθος των πρώτων αριθμών που είναι μικρότεροι από κάποιον δεδομένο αριθμό

Η ομιλία του Riemann μεταφρασμένη από τα Γερμανικά στα Αγγλικά από τον H. M. Edwards. Περιλαμβάνεται στο βιβλίο του Riemann's Zeta Function (Academic Press-1974)

APPENDIX

On the Number of Primes Less Than a Given Magnitude

by BERNHARD RIEMANN†

I believe I can best express my gratitude for the honor which the Academy has bestowed on me in naming me as one of its correspondents by immediately availing myself of the privilege this entails to communicate an investigation of the frequency of prime numbers, a subject which because of the interest shown in it by Gauss and Dirichlet over many years seems not wholly unworthy of such a communication.

In this investigation I take as my starting point the observation of Euler that the product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

where p ranges over all prime numbers and n over all whole numbers. The function of a complex variable s which these two expressions define when they converge I denote by $\zeta(s)$. They converge only when the real part of s is greater than 1; however, it is easy to find an expression of the function which always is valid. By applying the equation

$$\int_0^\infty e^{-nx}x^{s-1}\,dx = \frac{\Pi(s-1)}{n^s},$$

one finds first

$$\Pi(s-1)\zeta(s) = \int_0^\infty \frac{x^{s-1}\,dx}{e^x-1}.$$

If one considers the integral

$$\int \frac{(-x)^{s-1}\,dx}{e^x-1}$$

†Translated from *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* [R1, p. 145] by H. M. Edwards.

from $+\infty$ to $+\infty$ in the positive sense around the boundary of a domain which contains the value 0 but no other singularity of the integrand in its interior, then it is easily seen to be equal to

$$(e^{-\pi si} - e^{\pi si}) \int_0^\infty \frac{x^{s-1}\,dx}{e^x - 1},$$

provided that in the many-valued function $(-x)^{s-1} = e^{(s-1)\log(-x)}$ the logarithm of $-x$ is determined in such a way that it is real for negative values of x. Thus

$$2 \sin \pi s\, \Pi(s-1)\, \zeta(s) = i \int_\infty^\infty \frac{(-x)^{s-1}\,dx}{e^x - 1}$$

when the integral is defined as above.

This equation gives the value of the function $\zeta(s)$ for all complex s and shows that it is single-valued and finite for all values of s other than 1, and also that it vanishes when s is a negative even integer.

When the real part of s is negative, the integral can be taken, instead of in the positive sense around the boundary of the given domain, in the negative sense around the complement of this domain because in that case (when $\operatorname{Re} s < 0$) the integral over values with infinitely large modulus is infinitely small. But inside this complementary domain the only singularities of the integrand are at the integer multiples of $2\pi i$, and the integral is therefore equal to the sum of the integrals taken around these singularities in the negative sense. Since the integral around the value $n2\pi i$ is $(-n2\pi i)^{s-1}(-2\pi i)$, this gives

$$2 \sin \pi s\, \Pi(s-1)\, \zeta(s) = (2\pi)^s \sum n^{s-1}[(-i)^{s-1} + i^{s-1}],$$

and therefore a relation between $\zeta(s)$ and $\zeta(1-s)$ which, by making use of known properties of the function Π, can also be formulated as the statement that

$$\Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s)$$

remains unchanged when s is replaced by $1 - s$.

This property of the function motivated me to consider the integral $\Pi((s/2) - 1)$ instead of the integral $\Pi(s-1)$ in the general term of $\sum n^{-s}$, which leads to a very convenient expression of the function $\zeta(s)$. In fact

$$\frac{1}{n^s}\Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2} = \int_0^\infty e^{-nn\pi x} x^{(s/2)-1}\,dx;$$

so when one sets

$$\sum_1^\infty e^{-nn\pi x} = \psi(x),$$

it follows that

$$\Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s) = \int_0^\infty \psi(x) x^{(s/2)-1}\,dx$$

or, because

$$2\psi(x) + 1 = x^{-1/2}\left[2\psi\left(\frac{1}{x}\right) + 1\right] \qquad \text{(Jacobi, Fund., p. 184),}$$

that

$$\Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s) = \int_1^\infty \psi(x)x^{(s/2)-1}\,dx + \int_0^1 \psi\left(\frac{1}{x}\right)x^{(s-3)/2}\,dx$$
$$+ \frac{1}{2}\int_0^1 (x^{(s-3)/2} - x^{(s/2)-1})\,dx$$
$$= \frac{1}{s(s-1)} + \int_1^\infty \psi(x)(x^{(s/2)-1} + x^{-(1+s)/2})\,dx.$$

I now set $s = \frac{1}{2} + ti$ and

$$\Pi\left(\frac{s}{2}\right)(s-1)\pi^{-s/2}\zeta(s) = \xi(t)$$

so that

$$\xi(t) = \tfrac{1}{2} - (tt + \tfrac{1}{4})\int_1^\infty \psi(x)x^{-3/4}\cos(\tfrac{1}{2}t\log x)\,dx$$

or also

$$\xi(t) = 4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx}\,x^{-1/4}\cos\left(\frac{1}{2}t\log x\right)dx.$$

This function is finite for all finite values of t and can be developed as a power series in tt which converges very rapidly. Now since for values of s with real part greater than 1, $\log\zeta(s) = -\sum\log(1 - p^{-s})$ is finite and since the same is true of the other factors of $\xi(t)$, the function $\xi(t)$ can vanish only when the imaginary part of t lies between $\frac{1}{2}i$ and $-\frac{1}{2}i$. The number of roots of $\xi(t) = 0$ whose real parts lie between 0 and T is about

$$= \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}$$

because the integral $\int d\log\xi(t)$ taken in the positive sense around the domain consisting of all values whose imaginary parts lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T is (up to a fraction of the order of magnitude of $1/T$) equal to $[T\log(T/2\pi) - T]i$ and is, on the other hand, equal to the number of roots of $\xi(t) = 0$ in the domain multiplied by $2\pi i$. One finds in fact about this many real roots within these bounds and it is very likely that all of the roots are real. One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation.

If one denotes by α the roots of the equation $\xi(\alpha) = 0$, then one can express $\log\xi(t)$ as

$$\sum\log\left(1 - \frac{tt}{\alpha\alpha}\right) + \log\xi(0)$$

because, since the density of roots of size t grows only like $\log(t/2\pi)$ as t grows, this expression converges and for infinite t is only infinite like $t \log t$; thus it differs from $\log \xi(t)$ by a function of tt which is continuous and finite for finite t and which, when divided by tt, is infinitely small for infinite t. This difference is therefore a constant, the value of which can be determined by setting $t = 0$.

With these preparatory facts, the number of primes less than x can now be determined.

Let $F(x)$, when x is not exactly equal to a prime, be equal to this number, but when x is a prime let it be greater by $\frac{1}{2}$ so that for an x where $F(x)$ jumps

$$F(x) = \frac{F(x+0) + F(x-0)}{2}.$$

If one sets

$$p^{-s} = s\int_p^\infty x^{-s-1}\,dx, \qquad p^{-2s} = s\int_{p^2}^\infty x^{-s-1}\,dx, \qquad \ldots$$

in the formula

$$\log \zeta(s) = -\sum \log(1 - p^{-s}) = \sum p^{-s} + \tfrac{1}{2}\sum p^{-2s} + \tfrac{1}{3}\sum p^{-3s} + \cdots,$$

one finds

$$\frac{\log \zeta(s)}{s} = \int_1^\infty f(x)x^{-s-1}\,dx$$

when one denotes

$$F(x) + \tfrac{1}{2}F(x^{1/2}) + \tfrac{1}{3}F(x^{1/3}) + \cdots$$

by $f(x)$.

This equation is valid for every complex value $a + bi$ of s provided $a > 1$. But when in such circumstances

$$g(s) = \int_0^\infty h(x)x^{-s}\,d\log x$$

is valid, the function h can be expressed in terms of g by means of Fourier's theorem. The equation splits when h is real and when $g(a + bi) = g_1(b) + ig_2(b)$ into the two equations

$$g_1(b) = \int_0^\infty h(x)x^{-a}\cos(b\log x)\,d\log x,$$

$$ig_2(b) = -i\int_0^\infty h(x)x^{-a}\sin(b\log x)\,d\log x.$$

When both equations are multiplied by $[\cos(b\log y) + i\sin(b\log y)]\,db$ and integrated from $-\infty$ to $+\infty$, one finds in both cases that the right side is $\pi h(y)y^{-\alpha}$ so that when they are added and multiplied by iy^a

$$2\pi i h(y) = \int_{a-\infty i}^{a+\infty i} g(s)y^s\,ds,$$

where the integration is to be carried out in such a way that the real part of s remains constant.†

The integral represents, for a value of y where the function $h(y)$ has a jump, the middle value between the two values of h on either side of the jump. The function f was defined in such a way that it too has this property, so one has in full generality

$$f(y) = \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i} \frac{\log \zeta(s)}{s} y^s \, ds.$$

For $\log \zeta$ one can now substitute the expression

$$\frac{s}{2}\log \pi - \log(s-1) - \log \Pi\left(\frac{s}{2}\right) + \sum_{\alpha} \log\left[1 + \frac{(s-\frac{1}{2})^2}{\alpha\alpha}\right] + \log \xi(0)$$

found above; the integrals of the individual terms of this expression will not converge, however, when they are taken to infinity, so it is advisable to reformulate the equation as

$$f(x) = -\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i} \frac{d\dfrac{\log \zeta(s)}{s}}{ds} x^s \, ds$$

by integration by parts.

Since

$$-\log \Pi\left(\frac{s}{2}\right) = \lim\left[\sum_{n=1}^{m} \log\left(1 + \frac{s}{2n}\right) - \frac{s}{2}\log m\right]$$

for $m = \infty$ and therefore,

$$-\frac{d\dfrac{1}{s}\log \Pi\left(\dfrac{s}{2}\right)}{ds} = \sum_{1}^{\infty} \frac{d\dfrac{1}{s}\log\left(1 + \dfrac{s}{2n}\right)}{ds},$$

all of the terms in the expression for $f(x)$ except for the term

$$\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i} \frac{1}{ss}\log \xi(0) x^s \, ds = \log \xi(0)$$

take the form

$$\pm \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i} \frac{d\left[\dfrac{1}{s}\log\left(1 - \dfrac{s}{\beta}\right)\right]}{ds} x^s \, ds.$$

But

$$\frac{d\left[\dfrac{1}{s}\log\left(1 - \dfrac{s}{\beta}\right)\right]}{d\beta} = \frac{1}{(\beta - s)\beta}$$

†This argument is not quite correct. See the relevant note in Riemann's collected works [R1] (translator's note).

and, when the real part of s is greater than the real part of β,

$$-\frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\frac{x^s\,ds}{(\beta-s)\beta}=\frac{x^\beta}{\beta}=\int_\infty^x x^{\beta-1}\,dx$$

or

$$=\int_0^x x^{\beta-1}\,dx$$

depending on whether† the real part of β is negative or positive. Thus

$$\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i}\frac{d\left[\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)\right]}{ds}x^s\,ds$$

$$=-\frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)x^s\,ds$$

$$=\int_\infty^x\frac{x^{\beta-1}}{\log x}\,dx+\text{const}$$

in the first case and

$$=\int_0^x\frac{x^{\beta-1}}{\log x}\,dx+\text{const}$$

in the second case.

In the first case the constant of integration can be determined by taking β to be negative and infinite. In the second case the integral from 0 to x takes on two values which differ by $2\pi i$ depending on whether the path of integration is in the upper halfplane or in the lower halfplane; if the path of integration is in the upper halfplane, the integral will be infinitely small when the coefficient of i in β is infinite and positive, and if the path is in the lower halfplane, the integral will be infinitely small when the coefficient of i in β is infinite and negative. This shows how to determine the values of $\log[1-(s/\beta)]$ on the left side in such a way that the constants of integration drop out.

By setting these values in the expression for $f(x)$ one finds

$$f(x)=\mathrm{Li}(x)-\sum_\alpha\left[\mathrm{Li}(x^{(1/2)+\alpha i})+\mathrm{Li}(x^{(1/2)-\alpha i})\right]$$

$$+\int_x^\infty\frac{1}{x^2-1}\frac{dx}{x\log x}+\log\xi(0),$$

where‡ the sum $\sum_\alpha$ is over all positive roots (or all roots with positive real parts) of the equation $\xi(\alpha)=0$, ordered according to their size. It is possible, by means of a more exact discussion of the function ξ, easily to show that with this ordering of the roots the sum of the series

$$\sum_\alpha\left[\mathrm{Li}(x^{(1/2)+\alpha i})+\mathrm{Li}(x^{(1/2)-\alpha i})\right]\log x$$

†Note that this excludes the possibility Re $\beta=0$ and therefore does not apply to roots, if any, on the imaginary axis (translator's note).

‡Concerning the erroneous value of $\log\xi(0)$ in this formula, see Chapter 1 (translator's note).

is the same as the limiting value of

$$\frac{1}{2\pi i}\int_{a-bi}^{a+bi} \frac{d\frac{1}{s}\sum \log\left[1+\frac{(s-\frac{1}{2})^2}{\alpha\alpha}\right]}{ds} x^s \, ds$$

as b grows without bound; by a different ordering, however, it can approach any arbitrary real value.

From $f(x)$ one can find $F(x)$ by inverting

$$f(x) = \sum \frac{1}{n} F(x^{1/n})$$

to find

$$F(x) = \sum (-1)^{\mu} \frac{1}{m} f(x^{1/m}),$$

where m ranges over all positive integers which are not divisible by any square other than 1 and where μ denotes the number of prime factors of m.

If $\sum_\alpha$ is restricted to a finite number of terms, then the derivative of the expression for $f(x)$ or, except for a part which decreases very rapidly as x increases,

$$\frac{1}{\log x} - 2\sum_{\alpha} \frac{\cos(\alpha \log x)x^{-1/2}}{\log x}$$

gives an approximate expression for the density of primes + half the density of prime squares $+\frac{1}{3}$ the density of prime cubes, etc., of magnitude x.

Thus the known approximation $F(x) = \mathrm{Li}(x)$ is correct only to an order of magnitude of $x^{1/2}$ and gives a value which is somewhat too large, because the nonperiodic† terms in the expression of $F(x)$ are, except for quantities which remain bounded as x increases,

$$\mathrm{Li}(x) - \tfrac{1}{2}\mathrm{Li}(x^{1/2}) - \tfrac{1}{3}\mathrm{Li}(x^{1/3}) - \tfrac{1}{5}\mathrm{Li}(x^{1/5}) + \tfrac{1}{6}\mathrm{Li}(x^{1/6}) - \tfrac{1}{7}\mathrm{Li}(x^{1/7}) + \cdots.$$

In fact the comparison of $\mathrm{Li}(x)$ with the number of primes less than x which was undertaken by Gauss and Goldschmidt and which was pursued up to $x =$ three million shows that the number of primes is already less than $\mathrm{Li}(x)$ in the first hundred thousand and that the difference, with minor fluctuations, increases gradually as x increases. The thickening and thinning of primes which is represented by the periodic terms in the formula has also been observed in the counts of primes, without, however, any possibility of establishing a law for it having been noticed. It would be interesting in a future count to examine the influence of individual periodic terms in the formula for the density of primes. More regular than the behavior of $F(x)$ is the behavior of $f(x)$ which already in the first hundred is on average very nearly equal to $\mathrm{Li}(x) + \log \xi(0)$.

†Strictly speaking, the terms $\mathrm{Li}(x^{(1/2)+\alpha i})$ are not periodic but merely oscillatory (translator's note).