How Gödel Transformed Set Theory

Juliet Floyd and Akihiro Kanamori

Kurt Gödel (1906–1978), with his work on the constructible universe L, established the relative consistency of the Axiom of Choice and the Continuum Hypothesis. More broadly, he secured the cumulative hierarchy view of the universe of sets and ensured the ascendancy of first-order logic as the framework for set theory. Gödel thereby transformed set theory and launched it with structured subject matter and specific methods of proof as a distinctive field of mathematics. What follows is a survey of prior developments in set theory and logic intended to set the stage, an account of how Gödel marshaled the ideas and constructions to formulate L and establish his results, and a description of subsequent developments in set theory that resonated with his speculations. The survey trots out in quick succession the groundbreaking work at the beginning of a young subject.

Numbers, Types, and Well-Ordering

Set theory was born on that day in December 1873 when Georg Cantor (1845–1918) established that *the continuum is not countable*: There is no bijection between the natural numbers $\mathbf{N} = \{0, 1, 2, 3, \dots\}$ and the real numbers $\mathbf{R}$, since for any (countable) sequence of reals one can specify nested intervals so that any real in the intersection will not be in the sequence. Cantor soon investigated ways to define bijections between sets of reals and the like. He stipulated that two sets have the same *power* if there is a bijection between them, and, implicitly at first, that one set has a *higher* power than another if there is an injection of the latter into the first but no bijection. In an 1878 publication he showed that $\mathbf{R}$, the plane $\mathbf{R} \times \mathbf{R}$, and generally $\mathbf{R}^n$ are all of the same power, but there were still only the two infinite powers as set out by his 1873 proof. At the end of the publication Cantor asserted a dichotomy:

> Every infinite set of real numbers either is countable or has the power of the continuum.

This was the *Continuum Hypothesis* (CH) in its nascent context, and the *continuum problem*, to resolve this hypothesis, would become a major motivation for Cantor's large-scale investigations of infinite numbers and sets.

In his *Grundlagen* of 1883, Cantor developed the *transfinite numbers* and the key concept of *well-ordering*. The progression of transfinite numbers could be depicted, in his later notation, in terms of natural extensions of arithmetical operations:

$$0, 1, 2, \dots \omega, \omega + 1, \omega + 2, \dots \omega + \omega (= \omega \cdot 2), \\ \dots \omega \cdot 3, \dots \omega \cdot \omega (= \omega^2), \dots \omega^3, \dots \omega^\omega, \dots$$

A relation $\prec$ is a *well-ordering* of a set if and only if it is a strict linear ordering of the set such that every nonempty subset has a $\prec$-least element. Well-orderings carry the sense of sequential counting, and the transfinite numbers serve as standards for gauging well-orderings. Cantor called the set of natural numbers $\mathbf{N}$ the first number class (I) and

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Juliet Floyd is professor of philosophy at Boston University. Her email address is `jfloyd@bu.edu`.

Akihiro Kanamori is professor of mathematics at Boston University. His email address is `aki@math.bu.edu`.

the set of numbers whose predecessors are in bijective correspondence with (I) the second number class (II). The infinite numbers in the above display are all in (II). Cantor conceived of (II) as bounded above and showed that (II) itself is not countable. Proceeding upward, Cantor called the set of numbers whose predecessors are in bijective correspondence with (II) the third number class (III), and so on. Cantor then propounded a basic principle in the *Grundlagen*:

> "It is always possible to bring any *well-defined* set into the form of a *well-ordered* set."

Sets are to be well-ordered and thus to be gauged by his numbers and number classes. With this framework Cantor had transformed CH into the positive assertion that (II) *and* **R** *have the same power*. However, an emerging problem for Cantor was that he could not even *define* a well-ordering of **R**; the continuum, at the heart of mathematics, could not be easily brought into the fold of the transfinite numbers.

Almost two decades after his initial 1873 proof, Cantor in 1891 came to his celebrated *diagonal* argument. In various guises the argument would become fundamental in mathematical logic. Cantor himself proceeded in terms of functions, ushering collections of arbitrary functions into mathematics, but we cast his result as is done nowadays in terms of the power set $\mathcal{P}(x) = \{y \mid y \subseteq x\}$ of a set x. *For any set* x, $\mathcal{P}(x)$ *has a higher power than* x.

First, the function associating each $a \in x$ with $\{a\}$ is an injection: $x \to \mathcal{P}(x)$. Suppose now that F is any function: $x \to \mathcal{P}(x)$. Consider the "diagonal" set $d = \{a \in x \mid a \notin F(a)\}$. If d itself were a value of F, say $d = F(b)$, then we would have the contradiction: $b \in d$ if and only if $b \notin d$. Hence, F cannot be surjective.

Cantor had been shifting his notion of set to a level of abstraction beyond sets of real numbers and the like; the diagonal argument can be drawn out of the earlier argument, and the new result generalized the old since $\mathcal{P}(\mathbf{N})$ and **R** have the same power. The new result showed for the first time that there is a set of a higher power than **R**, e.g., $\mathcal{P}(\mathcal{P}(\mathbf{N}))$.

Cantor's *Beiträge* of 1895 and 1897 presented his mature theory of the transfinite. Cantor reconstrued power as *cardinal number*, now an autonomous concept beyond *une façon de parler* about bijective correspondence. He defined the addition, multiplication, and exponentiation of cardinal numbers primordially in terms of set-theoretic operations and functions. As befits the introduction of new numbers Cantor then introduced a new notation, one using the Hebrew letter aleph, $\aleph$. $\aleph_0$ is to be the cardinal number of **N** and the successive alephs

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$$

are now to be the cardinal numbers of the successive number classes from the *Grundlagen* and thus to exhaust all the infinite cardinal numbers. Cantor pointed out that $2^{\aleph_0}$ is the cardinal number of **R**, but frustrated in his efforts to establish CH he did not even mention the hypothesis, which could now have been stated as $2^{\aleph_0} = \aleph_1$. Every well-ordered set has an aleph as its cardinal number, but where is $2^{\aleph_0}$ in the aleph sequence?

CH was thus embedded in the very interstices of the beginnings of set theory. The structures that Cantor built, while now of great intrinsic interest, emerged largely out of efforts to articulate and establish it. The continuum problem was made the very first in David Hilbert's famous list of problems at the 1900 International Congress of Mathematicians; Hilbert drew out Cantor's difficulty by suggesting the desirability of "actually giving" a well-ordering of **R**.

Bertrand Russell (1872–1970), a main architect of the analytic tradition in philosophy, focused in 1900 on Cantor's work. Russell was pivoting from idealism toward a realism about propositions and with it logicism, the thesis that mathematics can be founded in logic. Taking a universalist approach to logic with all-encompassing categories, Russell took the class of all classes to have the largest cardinal number but saw that Cantor's 1891 result leading to higher cardinal numbers presented a problem. Analyzing that argument, by the spring of 1901 he came to the famous *Russell's Paradox*, a surprisingly simple counterexample to *full comprehension*, the assertion that for every property $A(x)$ the collection of objects having that property, the *class* $\{x \mid A(x)\}$, is also an object. Consider Russell's $\{x \mid x \notin x\}$. If this were an object r, then we would have the contradiction $r \in r$ if and only if $r \notin r$. Gottlob Frege (1848-1925) was the first to systematize quantificational logic in a formalized language, and he aimed to establish a purely logical foundation for arithmetic. Russell famously communicated his paradox to Frege in 1902, who immediately saw that it revealed a contradiction within his mature logical system.

Russell's own reaction was to build a complex logical structure, one used later to develop mathematics in Whitehead and Russell's 1910-3 *Principia Mathematica*. Russell's *ramified theory of types* is a scheme of logical definitions based on *orders* and *types* indexed by the natural numbers. Russell proceeded "intensionally"; he conceived this scheme as a classification of propositions based on the notion of *propositional function*, a notion not reducible to membership (extensionality). Proceeding in modern fashion, we may say that the universe of the *Principia* consists of *objects* stratified into disjoint types T_n, where T_0 consists of the *individuals*, $T_{n+1} \subseteq \{Y \mid Y \subseteq T_n\}$, and the types T_n for $n > 0$ are further ramified into orders O_n^i with

$T_n = \bigcup_i O_n^i$. An object in O_n^i is to be defined either in terms of individuals or of objects in some fixed O_m^j for some $j < i$ and $m \leq n$, the definitions allowing for quantification only over O_m^j. This precludes Russell's Paradox and other "vicious circles", as objects consist only of previous objects and are built up through definitions referring only to previous stages. However, in this system it is impossible to quantify over all objects in a type T_n, and this makes the formulation of numerous mathematical propositions at best cumbersome and at worst impossible. Russell was led to introduce his *Axiom of Reducibility*, which asserts that *for each object there is a predicative object consisting of exactly the same objects*, where an object is *predicative* if its order is the least greater than that of its constituents. This axiom reduced consideration to individuals, predicative objects consisting of individuals, predicative objects consisting of predicative objects consisting of individuals, and so on—the *simple theory of types*. In traumatic reaction to his paradox Russell had built a complex system of orders and types only to collapse it with his Axiom of Reducibility, a fearful symmetry imposed by an artful dodger.

Ernst Zermelo (1871–1953) made his major advances in set theory in the first decade of the new century. Zermelo's first substantial result was his independent discovery of the argument for Russell's Paradox. He then established in 1904 the Well-Ordering Theorem, that *every set can be well-ordered*, assuming what he soon called the Axiom of Choice (AC). Zermelo thereby shifted the notion of set away from Cantor's principle that every well-defined set is well-orderable and replaced that principle by an explicit axiom.

In retrospect Zermelo's argument for his Well-Ordering Theorem proved to be pivotal for the development of set theory. To summarize, suppose that x is a set to be well-ordered, and through Zermelo's AC hypothesis assume that the power set $\mathcal{P}(x) = \{y \mid y \subseteq x\}$ has a choice function, i.e., a function γ such that for every nonempty member y of $\mathcal{P}(x)$, $\gamma(y) \in y$. Call a subset y of x a *γ-set* if there is a well-ordering R of y such that for each $a \in y$, $\gamma(\{z \mid z\ R\ a \text{ fails}\}) = a$. That is, each member of y is what γ "chooses" from what does not R-precede it. The main observation is that γ-sets cohere in the following sense: If y is a γ-set with well-ordering R and z is a γ-set with well-ordering S, then $y \subseteq z$ and S is a prolongation of R, or vice versa. With this, let w be the union of all the γ-sets. Then w too is a γ-set, and by its maximality it must be all of x, and hence x is well-ordered.

Cantor's work had served to exacerbate a growing discord among mathematicians with respect to two related issues: whether infinite collections can be mathematically investigated at all, and how far the function concept is to be extended. The positive use of an arbitrary function operating on arbitrary subsets of a set having been made explicit, there was open controversy after the appearance of Zermelo's proof. This can be viewed as a turning point for mathematics, with the subsequent tilting toward the acceptance of AC symptomatic of a conceptual shift.

Axiomatization

In response to his critics Zermelo published a second proof of the Well-Ordering Theorem in 1908, and with axiomatization assuming a general methodological role in mathematics he also published in 1908 the first full-fledged axiomatization of set theory. But as with Cantor's work, this was no idle structure building, but a response to pressure for a new mathematical context. In this case it was not for the formulation and solution of a *problem* but rather to clarify a *proof*. Zermelo's motive in large part for axiomatizing set theory was to buttress his Well-Ordering Theorem by making explicit its underlying set existence assumptions.

To summarize Zermelo's axioms much as they would be presented today, there is an initial axiom asserting that two sets are the same if they contain the same members (Extensionality, i.e., membership determines equality), and an axiom asserting that there is an initial set $\emptyset$ having no members (Empty Set). Then there are the generative axioms, specific instances of comprehension: For any sets x, y, $\{x, y\} = \{z \mid z = x \text{ or } z = y\}$ is a set (Pairs), $\bigcup x = \{z \mid \exists y(y \in x \text{ and } z \in y)\}$ is a set (Union), and $\mathcal{P}(x) = \{y \mid y \subseteq x\}$ is a set (Power Set). There is an axiom asserting the existence of a particular recursively specified infinite set (Infinity). Zermelo aptly formulated AC in terms of sets as follows: *For any set x consisting of nonempty, pairwise disjoint sets, there is a set y such that each member of x intersects with y in exactly one element.* Finally, there is the axiom (schema) of Separation: *For any set x and "definite" property $A(y)$, $\{y \in x \mid A(y)\}$ is a set.* That is, the intersection of a set x and a class $\{y \mid A(y)\}$ is again a set. Zermelo saw that Separation suffices for a development of set theory that still allows for the "logical" formation of sets according to property; Russell's Paradox is precluded since only "logical" *sub*sets are to be allowed. But what exactly is a "definite" property? This was a central vagary that would be addressed in the subsequent formalization of Zermelo's set theory.

With his axioms Zermelo ushered in a new, abstract view of sets as structured solely by membership and built up iteratively according to governing axioms, a view that would soon come to dominate. Zermelo's work also pioneered the reduction of mathematical concepts and arguments to set-theoretic concepts and arguments from axioms, based on sets doing the work of mathematical objects.

Unlike the development of classical mathematics from marketplace arithmetic and Greek geometry, sets were neither laden with nor bolstered by well-worked antecedents. Zermelo axiomatization, unlike Russell's cumbersome theory of types, provided a simple system for the development of mathematics. Set theory would provide an underpinning of mathematics, and Zermelo's axioms would resonate with mathematical practice.

In the 1920s fresh initiatives structured the loose Zermelian framework with new features and corresponding developments in axiomatics, the most consequential moves made by John von Neumann (1903–1957) in his dissertation, with anticipations by Dimitry Mirimanoff (1861–1945). The transfinite numbers had been central for Cantor but peripheral to Zermelo, and in Zermelo's system not even $2^{\aleph_0} = \aleph_1$ could be stated directly. Von Neumann reconstrued the transfinite numbers as *bona fide* sets, the ordinals, and established their efficacy by formalizing transfinite recursion.

Ordinals manifest the idea, natural once iterative set formation is assimilated, of taking the relation of precedence in a well-ordering simply to be membership. A set (or class) x is *transitive* if and only if whenever $a \in b$ for $b \in x$, $a \in x$. A set x is a (von Neumann) *ordinal* if and only if x is transitive, and the membership relation restricted to $x = \{y \mid y \in x\}$ is a well-ordering of x. The first several ordinals are $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$, to be taken as the natural numbers $0, 1, 2, 3, \ldots$. The union of these finite ordinals is an ordinal, to be taken as ω; $\omega \cup \{\omega\}$ is an ordinal, to be taken as $\omega + 1$; and so forth. It has become customary to use the Greek letters $\alpha, \beta, \gamma, \ldots$ to denote ordinals; the class of all ordinals is itself well-ordered by membership, and $\alpha < \beta$ is written for $\alpha \in \beta$; and an ordinal without an immediate predecessor is a *limit* ordinal. Von Neumann established, as had Mirimanoff before him, the key instrumental property of Cantor's ordinal numbers for ordinals: *Every well-ordered set is order-isomorphic to exactly one ordinal with membership.* The proof was the first to make full use of the Axiom of Replacement and thus drew that axiom into set theory.

For a set x and property $A(v, w)$, the property is said to be *functional on* x if for any $a \in x$, there is exactly one b such that $A(a, b)$. The Axiom (schema) of Replacement asserts: *For any set* x *and property* $A(v, w)$ *functional on* x, $\{b \mid \exists a(a \in x \text{ and } A(a, b))\}$ *is a set.* This axiom posits sets that result when members of a set are "replaced" according to a property; a simple argument shows that Replacement subsumes Separation.

Von Neumann generally ascribed to the ordinals the role of Cantor's ordinal numbers, and already to incorporate transfinite arithmetic into set theory he saw the need to establish the Transfinite Recursion Theorem, the theorem that validates recursive definition along well-orderings. The proof had an antecedent in the Zermelo 1904 proof, but Replacement was necessary even for the very formulation, let alone the proof, of the theorem. With the ordinals in place von Neumann completed the incorporation of the Cantorian transfinite by defining the *cardinals* as the *initial ordinals*, those ordinals not in bijective correspondence with any of their predecessors.

Replacement has been latterly regarded as somehow less necessary or crucial than the other axioms, the purported effect of the axiom being only on large-cardinality sets. Initially, Abraham Fraenkel (1891–1965) and Thoralf Skolem (1887–1963) had independently proposed adjoining Replacement to ensure that $E(a) = \{a, \mathcal{P}(a), \mathcal{P}(\mathcal{P}(a)), \ldots\}$ would be a set for a, the infinite set given by Zermelo's Axiom of Infinity, since, as they pointed out, Zermelo's axioms cannot establish this. However, even $E(\emptyset)$ cannot be proved to be a set from Zermelo's axioms, and if his Axiom of Infinity were reformulated to accommodate $E(\emptyset)$, there would still be many finite sets a such that $E(a)$ cannot be proved to be a set. Replacement serves to rectify the situation by admitting new infinite sets defined by "replacing" members of the one infinite set given by the Axiom of Infinity. In any case, the full exercise of Replacement is part and parcel of transfinite recursion, which is now used everywhere in modern set theory, and it was von Neumann's formal incorporation of this method into set theory, as necessitated by his proofs, that brought in Replacement.

Von Neumann (and others) also investigated the salutary effects of restricting the universe of sets to the well-founded sets. The *well-founded* sets are the sets that belong to some "rank" V_α, these definable through transfinite recursion:

$$V_0 = \emptyset;\ V_{\alpha+1} = \mathcal{P}(V_\alpha);\ \text{and}\ V_\delta = \bigcup\{V_\alpha \mid \alpha < \delta\}\ \text{for limit ordinals}\ \delta.$$

$V_{\omega+1}$ contains every set consisting of natural numbers (finite ordinals), and so already at early levels there are set counterparts to many objects in mathematics. That the universe V of all sets is the *cumulative hierarchy*

$$V = \bigcup\{V_\alpha \mid \alpha \text{ is an ordinal}\}$$

is thus the assertion that every set is well-founded. Von Neumann essentially showed that this assertion is equivalent to a simple assertion about sets, the Axiom of Foundation: *Any nonempty set* x *has a member* y *such that* $x \cap y$ *is empty.* Thus, nonempty well-founded sets have $\in$-minimal members. If a set x satisfies $x \in x$, then $\{x\}$ is not well-founded; similarly, if there are $x_1 \in x_2 \in x_1$, then $\{x_1, x_2\}$ is not well-founded. Ordinals and sets consisting of ordinals are well-founded, and well-foundedness can be viewed as a generalization of the notion of being an ordinal that loosens the connection with transitivity. The Axiom of Foundation eliminates pathologies like $x \in x$ and through the cumulative hierarchy rendition allows inductive arguments to establish results about the entire universe.

In a remarkable 1930 publication Zermelo provided his final axiomatization of set theory, one that recast his 1908 axiomatization and incorporated both Replacement and Foundation. He herewith completed his transmutation of the notion of set, his abstract, prescriptive view stabilized by further axioms that structured the universe of sets. Replacement provided the means for transfinite recursion and induction, and Foundation made possible the application of those means to get results about *all* sets. Zermelo proceeded to offer a striking, synthetic view of a procession of natural models for his axioms that would have a modern resonance and applied Replacement and Foundation to establish isomorphism and embedding results.

Zermelo's 1930 publication was in part a response to Skolem's advocacy, already in 1922, of the idea of framing Zermelo's 1908 axioms in first-order logic. *First-order logic* is the logic of formal languages consisting of formulas built up from specified function and relation symbols using logical connectives and first-order quantifiers $\forall$ and $\exists$, quantifiers to be interpreted as ranging over the *elements* of a domain of discourse. (*Second-order logic* has quantifiers to be interpreted as ranging over arbitrary subsets of a domain.) Skolem had proposed formalizing Zermelo's axioms in the first-order language with $\in$ and $=$ as binary relation symbols. Zermelo's *definite* properties would then be those expressible in this first-order language in terms of given sets, and Separation would become a schema of axioms, one for each first-order formula. Analogous remarks apply to the formalization of Replacement in first-order logic. As set theory was to develop, the formalization of Zermelo's 1930 axiomatization in first-order logic would become the standard axiomatization, *Zermelo-Fraenkel with Choice* (ZFC). The "Fraenkel" acknowledges Fraenkel's early suggestion of incorporating Replacement. *Zermelo-Fraenkel* (ZF) is ZFC without AC.

Significantly, before this standardization both Skolem and Zermelo raised issues about the limitations of set theory as cast in first-order logic. Skolem had established a fundamental result for first-order logic with the Löwenheim-Skolem Theorem: *If a countable collection of first-order sentences has a model, then it has a countable model.* Having proposed framing set theory in first-order terms, Skolem pointed out as a palliative for taking set theory as a foundation for mathematics what has come to be called the *Skolem Paradox*: Zermelo's 1908 axioms when cast in first-order logic become a countable collection of sentences, and so if they have a model at all, they have a countable model. We thus have the "paradoxical" existence of countable models for Zermelo's axioms although they entail the existence of uncountable sets. Zermelo found this antithetical and repugnant, and proceeded in avowedly second-order terms in his 1930 work. However, stronger currents were at work leading to the ascendancy of first-order logic.

Constructible Universe

Enter Gödel. Gödel virtually completed the mathematization of logic by submerging "metamathematical" methods into mathematics. The Completeness Theorem from his 1930 dissertation established that logical consequence could be captured by formal proof for first-order logic and secured its key instrumental property of compactness for building models. The main advance was of course the direct coding, "the arithmetization of syntax", which together with a refined version of Cantor's diagonal argument led to the celebrated 1931 Incompleteness Theorem. This theorem established a fundamental distinction between what is *true* about the natural numbers and what is *provable* and transformed a program advanced by Hilbert in the 1920s to establish the consistency of mathematics by finitary means. Gödel's work showed in particular that for a (schematically definable) collection of axioms A, its *consistency*, that from A one cannot prove a contradiction, has a formal counterpart in an arithmetical formula Con(A) about natural numbers. Gödel's "second" theorem asserts that if A is consistent and subsumes the elementary arithmetic of the natural numbers, then Con(A) cannot be proved from A.

Gödel's advances in set theory can be seen as part of a steady intellectual development from his fundamental work on incompleteness. His 1931 paper had a prescient footnote 48a:

> As will be shown in Part II of this paper, the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite (cf. D. Hilbert, "Über das Unendliche", Math. Ann. **95**, p. 184), while in any formal system at most countably many of them are available. For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added (for example, the type ω to the system P [the simple theory of types superposed on the natural numbers as individuals satisfying the Peano

axioms]). An analogous situation prevails for the axiom system of set theory.

Gödel's letters and lectures clarify that the addition of an infinite type ω to Russell's theory of types would provide a "definition for 'truth' " for the theory and hence establish hitherto unprovable propositions like those provided by his Incompleteness Theorem. Inherent in Russell's theory was the indexing of types by the natural numbers, and Gödel's citation in the footnote of Hilbert's 1926 paper in connection with the possibility of adjoining transfinite types would bridge the past and the future. Hilbert there had attempted a proof of CH using transfinite indexing in his formalism, and Gödel would achieve what success is possible in this direction. Gödel never published the announced Part II, which was to have been on truth, but his engagement with truth and its distinction from provability could be viewed as his entrée into full blown set theory. In a 1933 lecture Gödel expounded on axiomatic set theory as a natural generalization of the simple theory of types "if certain superfluous restrictions" are removed: One could *cumulate* the types starting with individuals D_0 and taking $D_{n+1} = D_n \cup \mathcal{P}(D_n)$, and one could *extend* the process into the transfinite, mindful that for any type-theoretic system S a new proposition (e.g., Con(S)) becomes provable if to S is adjoined the "the next higher type and the axioms concerning it". Thus Gödel came to the cumulative hierarchy as a transfinite extension of the theory of types that incorporates higher and higher levels of truth.

Alfred Tarski (1902–1983) completed the mathematization of logic in the early 1930s by providing a systematic "definition of truth", exercising philosophers ever since. Tarski simply schematized truth by taking it to be a correspondence between formulas of a formal language and set-theoretic assertions about an interpretation of the language and by providing a recursive definition of the *satisfaction* relation, that which obtains when a formula holds in an interpretation. The eventual effect of Tarski's mathematical formulation of semantics would be not only to make mathematics out of the informal notion of satisfiability, but also to enrich ongoing mathematics with a systematic method for forming mathematical analogues of several intuitive semantic notions. For coming purposes, the following specifies notation and concepts:

For a first-order language, suppose that M is an interpretation of the language (i.e., a specification of a domain as well as interpretations of the function and relation symbols), $\varphi(v_1, v_2, \ldots, v_n)$ is a formula of the language with the (unquantified) variables as displayed, and $a_1, a_2, \ldots, a_n$ are the domain of M. Then

$$M \models \varphi[a_1, a_2, \ldots, a_n]$$

asserts that the formula φ is satisfied in M according to Tarski's recursive definition when v_i is interpreted as a_i. A subset y of the domain of M is *first-order definable over* M if and only if there is a formula $\psi(v_0, v_1, v_2, \ldots, v_n)$ and $a_1, a_2, \ldots, a_n$ in the domain of M such that

$$y = \{z \mid M \models \psi[z, a_1, \ldots, a_n]\}.$$

Set theory was launched on an independent course as a distinctive field of mathematics by Gödel's formulation of the class L of *constructible* sets through which he established the relative consistency of AC and CH. He thus attended to the fundamental issues raised at the beginning of set theory by Cantor and Zermelo. In his first 1938 announcement Gödel described L as a hierarchy "which can be obtained by Russell's ramified hierarchy of types, if extended to include transfinite orders." Indeed, with L Gödel had refined the cumulative hierarchy of sets to a cumulative hierarchy of *definable* sets which is analogous to the orders of Russell's *ramified* theory. Gödel's further innovation was to continue the indexing of the hierarchy through *all* the ordinals. Von Neumann's canonical well-orderings would be the spine for a thin hierarchy of sets, and this would be the key to both the AC and CH results. In a 1939 note Gödel informally presented L essentially as is done today: For any set x let def(x) denote the collection of subsets of x first-order definable over x according to the previous definition. Then define:

$$L_0 = \emptyset;\ L_{\alpha+1} = \mathrm{def}(L_\alpha),\ L_\delta = \bigcup\{L_\alpha \mid \alpha < \delta\}$$
$$\text{for limit ordinals } \delta;$$

and the *constructible universe*

$$L = \bigcup\{L_\alpha \mid \alpha \text{ is an ordinal}\}.$$

Gödel pointed out that L "can be defined and its theory developed in the formal systems of set theory themselves." This follows by transfinite recursion from the formalizability of def(x) in set theory, the "definability of definability", which was later reaffirmed by Tarski's systematic definition of the satisfaction relation in set-theoretic terms. In modern parlance, an *inner model* is a transitive class containing all the ordinals such that, with membership and quantification restricted to it, the class satisfies each axiom of ZF. Gödel in effect argued in ZF to show that L is an inner model and moreover that L satisfies AC and CH. He thus established the relative consistency Con(ZF) implies Con(ZFC + CH).

In his 1940 monograph, based on 1938 lectures, Gödel formulated L via a transfinite recursion that generated L set by set. His incompleteness proof had featured "Gödel numbering", the encoding of formulas by natural numbers, and his L recursion was a veritable Gödel numbering with ordinals, one that relies on their extent as given beforehand to generate a universe of sets. This approach may have obfuscated the satisfaction aspects of the construction, but on the other hand it did make more evident other aspects: Since there is a direct, definable well-ordering of L, choice functions abound in L, and AC holds there. Also, L was seen to have the important property of *absoluteness* through the simple operations involved in Gödel's recursion, one consequence of which is that for any inner model M, the construction of L in the sense of M again leads to the same class L. Decades later many inner models based on first-order definability would be investigated for which absoluteness considerations would be pivotal, and Gödel had formulated the canonical inner model, rather analogous to the algebraic numbers for fields of characteristic zero.

In a 1939 lecture about L Gödel described what amounts to the Russell orders for the simple situation when the members of a countable collection of real numbers are taken as the individuals and new real numbers are successively defined via quantification over previously defined real numbers, and he emphasized that the process can be continued into the transfinite. He then observed that this procedure can be applied to sets of real numbers and the like, as individuals, and moreover, that one can "intermix" the procedure for the real numbers with the procedure for sets of real numbers "by using in the definition of a real number quantifiers that refer to sets of real numbers, and similarly in still more complicated ways." Gödel called a constructible set "the most general [object] that can at all be obtained in this way, where the quantifiers may refer not only to sets of real numbers, but also to sets of sets of real numbers and so on, *ad transfinitum*, and where the indices of iteration …can also be arbitrary transfinite ordinal numbers." Gödel considered that although this definition of constructible set might seem at first to be "unbearably complicated", "the *greatest generality yields,* as it so often does, at the same time the *greatest simplicity*." Gödel was picturing Russell's ramified theory of types by first disassociating the types from the orders, with the orders here given through definability and the types represented by real numbers, sets of real numbers, and so forth. Gödel's intermixing then amounted to a recapturing of the complexity of Russell's ramification, the extension of the hierarchy into the transfinite allowing for a new simplicity.

Gödel went on to describe the universe of set theory, "the objects of which set theory speaks", as falling into "a transfinite sequence of Russellian [simple] types", the cumulative hierarchy of sets. He then formulated the constructible sets as an analogous hierarchy, the hierarchy of his 1939 note. In a comment bringing out the intermixing of types and orders, Gödel pointed out that "there are sets *of lower type* that *can* only *be defined* with the help of *quantifiers for sets of higher type*." For example, constructible members of $V_{\omega+1}$ in the cumulative hierarchy will first appear quite high in the constructible L_α hierarchy; resonant with Gödel's earlier remarks about truth, members of $V_{\omega+1}$, in particular sets of natural numbers, will encode truth propositions about higher L_α's. Gödel had given priority to the ordinals and recursively formulated a hierarchy of orders based on definability, and the hierarchy of types was spread out across the orders. The jumble of the *Principia Mathematica* had been transfigured into the constructible universe L.

Gödel's argument for CH holding in L rests, as he himself wrote in a brief 1939 summary, on "a generalization of Skolem's method for constructing enumerable models", now embodied in the well-known Skolem Hull argument and Condensation Lemma for L. It is the first significant application of the Löwenheim-Skolem Theorem since Skolem's own to get his paradox. Ironically, though Skolem sought through his paradox to discredit set theory based on first-order logic as a foundation for mathematics, Gödel turned paradox into method, one promoting first-order logic. Gödel showed that in L every subset of L_α belongs to some L_β for some β of the same power as α (so that in L every real belongs to some L_β for a countable β, and CH holds). In the 1939 lecture he asserted that "this fundamental theorem constitutes the corrected core of the so-called Russellian axiom of reducibility." Thus, Gödel established another connection between L and Russell's ramified theory of types. But while Russell had to *postulate* his Axiom of Reducibility for his finite orders, Gödel was able to *derive* an analogous form for his transfinite hierarchy, one that asserts that the types are delimited in the hierarchy of orders.

Gödel brought into set theory a method of construction and argument and thereby affirmed several features of its axiomatic presentation. First, Gödel showed how first-order definability can be formalized and used in a transfinite recursive construction to establish striking new mathematical results. This significantly contributed to a lasting ascendancy for first-order logic which beyond its *sufficiency* as a logical framework for mathematics was seen to have considerable *operational efficacy.* Gödel's construction moreover buttressed the incorporation of Replacement and Foundation into set theory. Replacement was immanent in the arbitrary extent of the ordinals for the indexing of L and in its formal definition via transfinite recursion. As for Foundation, underlying the construction was the well-foundedness of sets. Gödel in a footnote to his 1939 note wrote: "In order to give A [the axiom $V = L$, that the universe is L] an intuitive meaning, one has to understand by 'sets' all objects obtained by building up the simplified hierarchy of types on an empty set of individuals (including types of arbitrary transfinite orders)." Some have been baffled about how the cumulative hierarchy picture came to dominate in set-theoretic practice; although there was Mirimanoff, von Neumann, and especially Zermelo, the picture came in with Gödel's method, the reasons being both thematic and historical: Gödel's work with L with its incisive analysis of first-order definability was readily recognized as a signal advance, while Zermelo's 1930 paper with its second-order vagaries remained somewhat obscure. As the construction of L was gradually digested, the sense that it promoted of a cumulative hierarchy reverberated to become the basic picture of the universe of sets.

New Axioms

How Gödel transformed set theory can be broadly cast as follows: On the larger stage, from the time of Cantor, sets began making their way into topology, algebra, and analysis so that by the time of Gödel, they were fairly entrenched in the structure and language of mathematics. But how were sets viewed among set *theorists*, those investigating sets as such? Before Gödel, the main concerns were what sets *are* and how sets and their axioms can serve as a reductive basis for mathematics. Even today, those preoccupied with ontology, questions of mathematical existence, focus mostly upon the set theory of the early period. After Gödel, the main concerns became what sets *do* and how set theory is to advance as an autonomous field of mathematics. The cumulative hierarchy picture was in place as subject matter, and the metamathematical methods of first-order logic mediated the subject. There was a decided shift toward epistemological questions, e.g., what can be proved about sets and on what basis.

As a pivotal figure, what was Gödel's own stance? What he *said* would align him more with his predecessors, but what he *did* would lead to the development of methods and models. In a 1944 article on Russell's mathematical logic, in a 1947 article on Cantor's continuum problem (and in a 1964 revision), and in subsequent lectures and correspondence, Gödel articulated his philosophy of "conceptual realism" about mathematics. He espoused a staunchly objective "concept of set" according to which the axioms of set theory are true and are descriptive of an objective reality schematized by the cumulative hierarchy. Be that as it may, his actual mathematical work laid the groundwork for the development of a range of models and axioms for set theory. Already in the early 1940s Gödel worked out for himself a possible model for the negation of AC, and in a 1946 address he described a new inner model, the class of ordinal definable sets.

In his 1947 article on the continuum problem Gödel pointed out the desirability of establishing the independence of CH, i.e., in addition to his relative consistency result, that also Con(ZF) implies Con(ZFC + the negation of CH). However, Gödel stressed that this would not *solve* the problem. The axioms of set theory do not "form a system closed in itself", and so the "very concept of set on which they are based" suggests their extension by new axioms, axioms that may decide issues like CH. New axioms could even be entertained on the extrinsic basis of the "fruitfulness of their consequences". Gödel concluded by advancing the remarkable opinion that CH "will turn out to be wrong" since it has as paradoxical consequences the existence of thin, in various senses he described, sets of reals of the power of the continuum.

Later touted as his "program", Gödel's advocacy of the search for new axioms mainly had to do with *large cardinal* axioms. These postulate structure in the higher reaches of the cumulative hierarchy, often by positing cardinals whose properties entail their inaccessibility from below in strong senses. Speculations about large cardinal possibilities had occurred as far back as the time of Zermelo's first axiomatization of set theory. Gödel advocated their investigation, and they can be viewed as a further manifestation of his footnote 48a idea of capturing more truth, this time by positing strong closure points for the cumulative hierarchy. In the early 1960s large cardinals were vitalized by the infusion of model-theoretic methods, which established their central involvement in *embeddings* of models of set theory. The subject was then to become a mainstream of set theory after the dramatic introduction of a new way of getting *extensions* of models of set theory.

Paul Cohen (1934–) in 1963 established the independence of AC from ZF and the independence of CH from ZFC. That is, Cohen established that Con(ZF) implies Con(ZF + the negation of AC) and that Con(ZF) implies Con(ZFC + the negation of CH). These results delimited ZF and ZFC in terms of the two fundamental issues raised at the beginning of set theory. But beyond that, Cohen's proofs were soon to flow into method, becoming the inaugural examples of *forcing*, a remarkably general and flexible method for extending models of set theory by adding "generic" sets. Forcing has strong intuitive underpinnings and reinforces the notion of set as given by the first-order ZF axioms with conspicuous uses of Replacement and Foundation. With L analogous to the field of algebraic numbers, forcing is analogous to making transcendental field extensions. If Gödel's construction of L had launched set theory as a distinctive field of mathematics, then Cohen's method of forcing began its transformation into a modern, sophisticated one. Set theorists rushed in and were soon establishing a cornucopia of relative consistency results, truths in a wider sense, some illuminating problems of classical mathematics. In this sea change the extent and breadth of the expansion of set theory dwarfed what came before, both in terms of the numbers of people involved and the results established.

Already in the 1960s and into the 1970s large cardinal postulations were charted out and elaborated, investigated because of the "fruitfulness of their consequences" since they provided quick proofs of various strong propositions and because they provided the consistency strength to establish new relative consistency results. A subtle connection quickly emerged between large cardinals and combinatorial propositions low in the cumulative hierarchy: Forcing showed just how relative the Cantorian notion of cardinality is, since bijections could be adjoined easily, often with little disturbance to the universe. In particular, large cardinals, highly inaccessible from below, were found to satisfy substantial propositions even after they were "collapsed" by forcing to $\aleph_1$ or $\aleph_2$, i.e., bijections were adjoined to make the cardinal the first or second uncountable cardinal. Conversely, such propositions were found to entail large cardinal hypotheses in the clarity of an L-like inner model, sometimes the very same initial large cardinal hypothesis. In a subtle synthesis, hypotheses of length concerning the extent of the transfinite were correlated with hypotheses of width concerning the fullness of power sets low in the cumulative hierarchy, sometimes the arguments providing *equiconsistencies*. Thus, large cardinals found not only extrinsic but intrinsic justifications. Although their emergence was historically contingent, large cardinals were seen to form a *linear* hierarchy, and there was the growing conviction that this hierarchy provides *the* hierarchy of exhaustive principles against which all possible consistency strengths can be gauged, a kind of hierarchical completion of ZFC.

In the 1970s and 1980s possibilities for new complementarity were explored with the development of *inner model theory* for large cardinals, the investigation of minimal L-like inner models having large cardinals, models that exhibited the kind of fine structure that Gödel had first explored for L. Also, *determinacy* hypotheses about sets of reals were explored because of their fruitful consequences in descriptive set theory, the definability theory of the continuum. Then in a grand synthesis, certain large cardinals were found to provide just the consistency strength to establish the consistency of $\mathrm{AD}^{L(\mathbf{R})}$, the Axiom of Determinacy holding in the minimal inner model $L(\mathbf{R})$ containing all the reals. In a different direction, Harvey Friedman has recently provided a variety of propositions of finite combinatorics that are equi-consistent with the existence of large cardinals; this incisive work serves to affirm the "necessary use" of large cardinal axioms even in finite mathematics. In set theory itself, Hugh Woodin has developed a scheme based on a new logic in an environment of large cardinals that argues against CH itself, and with an additional axiom, that $2^{\aleph_0} = \aleph_2$. These results serve as remarkable vindications for Gödel's original hopes for large cardinals.

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